Method and Algorithm for Interval Maximum Expected Flow in a Network

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***Key Words**:* Interval expected flow; interval probability; smallest capacity, uncertainty.

Abstract. This paper deals with interval maximum expected flow in a capacitated network, when each arc has two parameters: interval probabilities, and capacity. The interval probabilities are uncertain, but lower and upper limits, within which they are expected to fall, are given. A method and algorithm are proposed to determine the maximum interval expected flow in the network.

1. Introduction

The Maximum Flow Problem (*MFP*) is one of the most fundamental problems in network flow theory. It is a network optimization problem.

Consider a *directed* and *connected* network G, $G = (N, A)$, having a source node, s, a destination node, t, and s, $t \in N$, and capacitated arcs $(i, j) \in A$. The cardinality of N and A are denoted by $|N|$ and $|A|$, respectively, and $|N| = n$, $|A| = m$.

The MFP is to find a pattern of flows through the arcs of A which will result in a maximum flow shipped from a source node to a destination node. It is assumed that the source node has an infinite amount of flow, and that the flow conservation law (condition) holds for all the intermediate nodes of the network [7,11,12]. Ford and Fulkerson in 1956 solved the MFP using the well known Augmenting Route Algorithm [3].

The basic idea of the classical maximum flow algorithm is to find a route from a source node to a destination node that has positive flow capacities. The maximum flow along this route must then be equal to the *smallest capacity*, c^*, among the capacities of all arcs comprising the route. Then, the maximum flow equals the sum of the c^*-values determined in the successive iterations [7,15].

The Augmenting Route is a directed route from the source node to the destination node in the residual network such that each arc on this route must have positive flow capacity. The minimum of these residual capacities is denoted as the residual capacities of the augmenting route because it represents the amount of flow that can feasibly be added to the entire route [7]. The augmenting route algorithm is the most intuitive maximum flow algorithm. It keeps finding augmenting routes to increase the total flow from source node to destination node until no more augmenting routes left [7,16].

A simple sequential algorithm for the MFP is proposed in [1], on a network with n nodes, m arcs, and integer arc capacities bound by U, and the running time of the algorithm is $O(nm+n^2\log U)$.

A new maximum flow algorithm is proposed in [14]. The running time of algorithm is $O(nm\log(U/n))$. If $U = O(n)$ then this algorithm runs in $O(nm)$ time for all values of m and n. It is a two-phase capacity scaling algorithm that combines other known methods: generic augmenting route algorithm, shortest augmenting route algorithm, and capacity scaling algorithm.

In [4], the author proposes an algorithm for MFP, and the running time of the algorithm is $O(n(m+n\log n)\log nU)$. A scaling version of this algorithm runs in $O(mn\log U)$ time. The scaling algorithm does not use any sophisticated data structures, like Fibonacci heap.

An insightful algorithm for the MFP is proposed in [13], when the parameters of a given network are not known exactly. It is a probabilistic approach which is based on the classical maximum flow algorithm. The corresponding problem is called the *maximum expected flow* problem. The author considers less degree of uncertainty, i.e., the given probability for each arc is certain. The dynamic programming is used to find the most reliable route and the reliability of this route. The author uses the concept of flow augmenting routes. This flow augmenting routes may increase or decrease the expected flow from the source node to the destination node (or it may unchanged the expected flow), since some routes will have capacities reduced and other routes will have capacities increased.

A parametric maximum flow algorithm for bipartite networks is presented in [2]. A bipartite network G, $G = (N, A)$, is a network where $N = N_1 \cup N_2$, $A \subseteq N_1 \times N_2$, $n_1 = |N_1|$, $n_2 = |N_2|$ and $n_1 \le n_2$. The arcs capacities are not fixed values but are functions of a single parameter λ, λ is a continuous real variable. The problem is to determine the minimum value of λ such that the maximum flow value in the corresponding network equals a given threshold. The time complexity of the algorithm is $O(n_1 n_2^2 \log(n_2/n_1))$.

A simultaneous parametric maximum-flow (*SPMF*) algorithm is given in [17]. The new algorithm does not use preflows. No labels of the nodes are used. This algorithm is not augmenting route algorithm. Two new algorithms SPMF^{simple} and SPMF^{fast} for the simultaneous parametric maximum flow are proposed in [18,19]. The SPMF^{simple} algorithm also works with a non-parametric network.

The paper is organized as follows. In section 2 the interval maximum expected flow method and algorithm are proposed, and two examples are considered. The conclusions are given in section 3.

2. Interval Maximum Expected Flow Method and Algorithm

The problem of determining the maximum flow that can be transmitted from a source node to a destination node in a network is proposed in [3].

In this section, we consider interval maximum expected flow problem. The aim of this problem is, for example, to develop a passenger schedule that maximizes the number of passengers sent between a source node and a destination node in a capacitated network, when the probabilities of arcs capacities are uncertain [9,10]. In real situation, it may happen that certain passenger schedule flights do not leave on time due to some technical reasons.

The main differences between the traditional maximum flow algorithm and the interval maximum expected flow algorithm are as follows: in the traditional case we select a route with **positive flow capacity**, whereas, in the interval expected flow algorithm we choose the route with **maximum reliability** and **positive flow capacity**.

Consider a capacitated network G, $G = (N, A)$, in which each arc (i, j) has two numbers: π_{ij} and c_{ij}. $\pi_{ij} = [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ is the interval probabilities that the flight between node i and node j leaves on time. The probability is uncertain, and given by its upper and lower limits. c_{ij} is the capacity of an arc (i, j), for example, the maximum number of passengers that can be carried along arc (i, j) during a time period. The capacity, c_{ij}, along arc (i, j) will be considered *zero* if the flight does not leave on time with probability $(1- \pi_{ij})$.

The reliability P of the route H from source node to the destination node, that is, the probabilities that the route H can be completed, is obtained by using the most reliable route interval algorithm based on '$\times$' operator, which is given in [5,6, 8,10].

Assume that an arc (i, j) has initial capacities $(\overset{\approx}{c}_{ij}, \overset{\approx}{c}_{ji})$. As portions of the initial capacities are committed to flow in the arc, the residuals (or remaining capacities) of the arc are updated. We use the notation (c_{ij}, c_{ji}), to represent these residuals. At the beginning we consider initial residuals (c_{ij}, c_{ji}), equals to the initial capacities $(\overset{\approx}{c}_{ij}, \overset{\approx}{c}_{ji})$ [15].

For a node j that receives flow from node i, we attach a label $[c_j, i]$, where c_j is the flow from node i to node j.

The flow in a network can be found, assuming perfect reliability, by assigning the numbers x_{ij} to various arcs (i, j), and $0 \le x_{ij} \le c_{ij}$ such that $\sum_i x_{ij} = \sum_k x_{jk}$ for all intermediate nodes j, i.e., an essential assumption in the MFP is based on the law of conservation (the total flow into a node must equals the total flow out of the node, for example, node j is neither the source node nor the destination node, the total amount of flow into node j must be equal to the total amount flow out from node j) [12].

Let P_j be the interval probabilities from node 1 to node j, $P_j = [\underline{p}_j, \overline{p}_j]$, and let π_{ij} be the interval probabilities between current node j and its predecessor i, and $\pi_{ij} = [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$, $i \in N_j$, where i ranges over the set of all preceding nodes N_j.

The most reliable route interval algorithm based on '$\times$' operator [5,6,8,10] is used to find the first most reliable route, the second most reliable route, ..., the z^{th} most reliable routes, H_1, H_2, ...,H_z, and the corresponding reliabilities, where z is the last most reliable route with positive flow.

Assume the first most reliable route H_1 between the source node and destination node has capacity C_1, the second most reliable route H_2 has capacity C_2, and so on. The feasible flow under perfect reliability is $(C_1 + C_2 + C_3 + ...)$. In our case, with imperfect reliability, the capacities C_i are random variables, and we have an expected total flow $E(C_1 + C_2 + C_3 + ...)$. From elementary probability theory, if C_1, C_2, etc. are independent of each other or not, we have $E(C_1 + C_2 + ...)= E(C_1) + E(C_2) + ...$

Let P_b be the reliabilities of the b^{th} most reliable route H_b from the source node 1 to the destination node n. Then, $P_b = [\underline{p}_b, \overline{p}_b] = \prod_{(i,j)\in H_b} [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$, $b = 1, 2, ..., z$.

The maximum flow along the most reliable route must be equal to the smallest capacity c_b^* of all arcs comprising the route H_b.

(1) $c_b^* = \{\min_{(i,j)\in H_b}(c_{ij})\}$.

The Interval Expected Capacity (IEC) of the route H_b is given by $IEC(H_b) = P_b \times c_b^* + (1 - P_b) \times 0 = P_b \times c_b^*$, the reliability P_b multiplied by the smallest capacity, c_b^*, of the route H_b. Hence,

(2) $IEC(H_b) = [\underline{p}_b, \overline{p}_b] \times c_b^*$.

The Total Maximum Flow (TMF) is the sum of all smallest capacities of the routes from source to sink. So,

(3) $TMF = c_1^* + c_2^* + ...+ c_z^*$.

The Interval Expected Flow (IEF) is the sum of all interval expected capacities ($IEC(H_i)$) of the routes H_i, $i = \overline{1,z}$ from source to sink. So,

(4) $IEF = IEC(H_1) + ... + IEC(H_z)$.

Let b denote the rank of reliability of the route H_b. Denote by AR the set of flow augmenting routes.

Set $AR = \varnothing$.

Set $b = 1$, $z = 0$, $IEF = TMF = 0$.

Set $G_1 = G = (N, A)$.

The algorithm consists of the following generalized steps:

***Step* 1**. Determine the b^{th} most reliable route H_b in the network G_1 and the maximum reliability P_b on this route by using the most reliable route interval algorithm, and using only real (non artificial) capacities in the directions of the committed flows. If no route with the positive flow exists, go to *step* 7, else go to *step* 2.

***Step* 2**. Obtain the smallest capacity c_b^* of the b^{th} most reliable route H_b by using (1).

***Step* 3**. Obtain the interval expected capacity $IEC(H_b)$ by using (2).

***Step* 4**. Change the current capacities (c_{ij}, c_{ji}) and probabilities of the arc (i, j), on the route H_b, and $\forall (i, j) \in H_b$.

Set
($c_{ij} = c_{ij} - c_b^*$, $c_{ji} = c_{ji} + c_b^*$) if the flow is from i to j.
($c_{ij} = c_{ij} + c_b^*$, $c_{ji} = c_{ji} - c_b^*$) if the flow is from j to i.
$\pi_{ij} = 0$ if $c_{ij} = 0$.

Comment. At least one arc of the most reliable route will have a capacity zero. This will ensure to find the next most reliable route with positive flow. We therefore have a new network G^*.

***Step* 5**. Determine the new network G^* after *step* 4.
$G^* = (N, A^*)$, where $A^* = \{(i, j)|(i, j) \in A, c_{ij} > 0\}$.
Set $G_1 = G^*$.

***Step* 6**. Find the TMF and IEF accumulated up to now.
Set $TMF = TMF + c_b^*$,
$IEF = IEF + IEC(H_b)$.
Set $z = z + 1$, $b = b + 1$.
Go to *step* 1.

***Step* 7**. Find a flow augmenting route H_a in the network G_1, such that $H_a \neq H_{ar}$, $H_{ar} \in AR$, and using artificial capacities. If no flow augmenting route with positive flow exists, go to *step* 13, else go to *step* 8.

***Step* 8**. Obtain the smallest capacity c_a^* of the most reliable route H_a.

***Step* 9**. Obtain the interval expected capacity $IEC(H_a)$.

***Step* 10**. Change the current capacities (c_{ij}, c_{ji}) of the arcs (i, j) on the route H_a, $\forall$ $(i, j) \in H_b$. Set
($c_{ij} = c_{ij} - c_a^*$, $c_{ji} = c_{ji} + c_a^*$) if the flow is from i to j.
($c_{ij} = c_{ij} + c_a^*$, $c_{ji} = c_{ji} - c_a^*$) if the flow is from j to i.
$\pi_{ij} = 0$ if $c_{ij} = 0$.

***Step* 11**. Correct the smallest capacities c_i^* of the most reliable routes H_i, $i \in \{1, 2, \ldots, z\}$ influenced by the route H_a.

Set $\tilde{c}_i^* = c_i^* + c_a^*$ if the smallest capacities of H_i is increased else set $\tilde{c}_i^* = c_i^* - c_a^*$ if the smallest capacities of H_i is decreased.

Set $\tilde{c}_i^* = c_i^*$ if the smallest capacities of H_i remains unchanged.

Determine the new total maximum flow ($NTMF$),

$$NTMF = \sum_{i=1}^{z} \tilde{c}_i^* .$$

Determine the new (corrected) interval expected capacities ($NIEC$), of the routes H_i, $NIEC(H_i) = P_i \times \tilde{c}_i^*$, $i = \overline{1, z}$.

Obtain the new interval expected flow ($NIEF$), from node 1 to node n, $NIEF = \sum_{i=1}^{z} NIEC(H_i)$.

If $NIEF > IEF$ go to *step* 12, else go to *step* 13.

Comment. A flow augmenting route will increases the flow from the source node to the destination node to a maximum. This may increase or decrease the interval expected flow from the source node to the destination node (it may also be unchanged), since some routes will reduced the capacities and other routes will increased the capacities. So, the sum of the interval expected capacities can be increased or decreased. The algorithm retains flow augmenting routes for which the interval expected flow is increased.

***Step* 12**. Determine the new network $G^* = (N, A^*)$, where $A^* = \{(i, j)|(i, j) \in A, c_{ij} > 0\}$.
Set $G_1 = G^*$, $IEF = NIEF$.
Set $AR = AR \cup H_a$.
Go to *step* 7.

***Step* 13**. Compute the optimal flow OF_{ij} in arc $(i, j) \in A$ as follows:

$$(\alpha, \beta) = (\overset{\approx}{c}_{ij} - c_{ij}, \overset{\approx}{c}_{ji} - c_{ji})$$

where $(\overset{\approx}{c}_{ij}, \overset{\approx}{c}_{ji}$ are the initial capacities, and (c_{ij}, c_{ji}) are the final capacities of arc (i, j).
If $\alpha > 0$, set $OF_{ij} = \alpha$.
If $\beta > 0$, set $OF_{ij} = \beta$.
Repeat *step* 13 for all $(i, j) \in A$.

Comment. It is not possible to have both α and β positive.

The convergence of the algorithm to the maximum interval expected flow is proven in the following theorems, which represent an interval generalization of the corresponding results in [13].

Theorem 1 If a maximum flow is obtained by steps 1 to 6 in the algorithm, then the maximum interval expected flow is the sum of the interval expected capacities of each route obtained.

Proof. There will not be any flow augmenting route that can increase the flow, if the flow obtained by steps 1 to 6 is maximum. Assume the interval expected flow obtained by summing the interval expected capacities is not maximum. Then by reducing the capacity of a route with lower reliability and increasing by same or less amount the capacity of a route with high reliability, the value of the obtained interval expected flow may be improved. But such increasing of the capacity of a route with higher reliability is not possible, because its capacity is zero. Hence, the value obtained for the interval expected flow must be the maximum value.

Theorem 2. The sum of the interval expected capacities of all routes obtained using steps 1 to 12 in the algorithm is the value of the maximum interval expected flow.

Proof. Assume that a flow augmenting route can increase the flow of the network to the maximum. The routes that are selected first are those with higher reliability, and their capacities may occupy some of the capacities of arcs of a route with lower reliability. Hence the flow augmenting route can only reduce capacity of one or more routes with higher reliability, and increase the capacity of two or more routes with lower reliability. Therefore, if the flow augmenting route makes an increase in the interval expected flow, then this interval expected flow is the maximum since a higher value that would correspond to a reduction of the capacity of a route with lower reliability and to an increase of the capacity of a route with higher reliability, is not possible. If the flow augmenting route gives a decrease of the interval expected flow, or leaves it unchanged, then the sum of the interval expected capacity of each route obtained previously is the maximum interval expected flow.

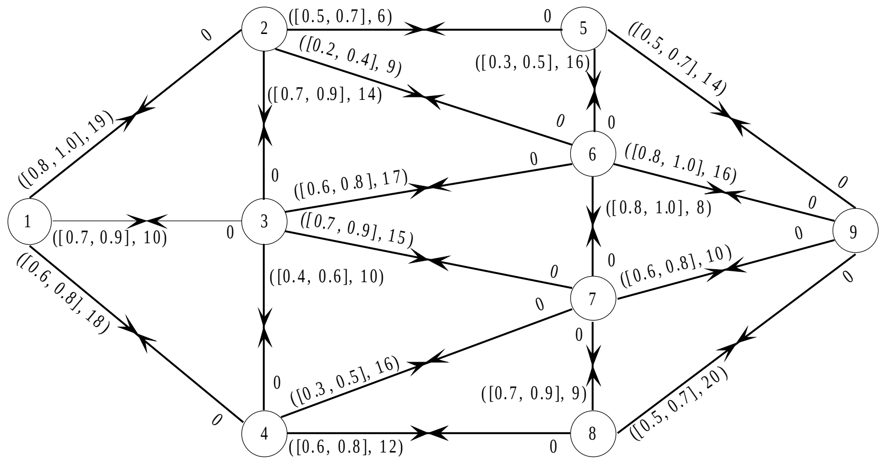

Figure 1. Network with interval probabilities and capacity

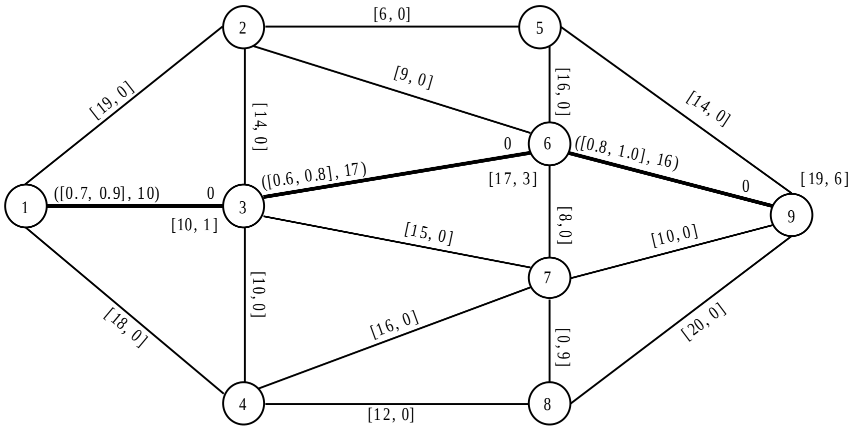

Figure 2. Network with the first most reliable route is (1 - 3 - 6 - 9), the reliability of this route is [0.336, 0.72], and $c_1^* = 10$

2.1. Numerical Example

Consider the network described in *figure 1*, where an arc (i, j) has two numbers: $\pi_{ij} = [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ is the interval probabilities that flight will leave on time, and c_{ij} is the capacity or the maximum number of passengers that can be carried along the arc (i, j).

Using the new algorithm, we can find the maximum interval expected flow from the source node to the destination node.

Iteration 1. First find the most reliable route on the given network using the most reliable route interval algorithm. The most reliable route H_1 is shown in *figure 2*, and denoted by „thick line". The most reliable route H_1 is (1 – 3 – 6 – 9), and the reliability of this route is [0.336, 0.72]. The minimum capacity along this route $c_1^* = 10$. The $IEC(H_1)$ on this route is [3.36, 7.20]. Now modify the capacity of all arcs on the route H_1. The residual capacities are:

$(c_{13}, c_{31}) = (10 - 10, 0 + 10) = (0, 10)$;

$(c_{36}, c_{63}) = (17 - 10, 0 + 10) = (7, 10)$;

$(c_{69}, c_{96}) = (16 - 10, 0 + 10) = (6, 10)$, and form a new network.

In a similar way the results of *iteration* 2 to *iteration* 6 are obtained.

No new route is found in the new network (after iteration 6) with positive capacity. So, we will move on *step* 6.

The total maximum flow (TMF), and the interval expected flow (IEF) are as follows:

$TMF = c_1^* + c_2^* + c_3^* + c_4^* + c_5^* + c_6^*$
$= 10 + 6 + 8 + 5 + 12 + 2 = \mathbf{43}$;

$IEF = [3.36, 7.20] + [1.613, 4.30] + [1.882, 5.184] + [1.00, 2.45] + [2.16, 5.376] + [0.216, 0.64] = \mathbf{[10.231, 25.170]}$.

Iteration 7. Now we will move to *step* 7, and using the „artificial capacities", a flow augmenting route is found in *figure 3*. It is (1 – 4 – 7 – 3 – 2 – 5 – 7), and the *smallest capacity* is 1. Hence, arcs (1 – 4), (4 – 7), (2 – 5) and (5 – 9) have capacities in $(i \to j)$ direction (in forward direction), but arcs (2 – 3) and (3 – 7) are used in $(j \to i)$ direction (in backward direction).

We therefore obtain the route (1 – 2 – 3 – 7 – 9), with higher reliability [0.2352, 0.6480], having its capacity reduced by one to a value of 7, and two routes (1 – 4 – 7 – 9)

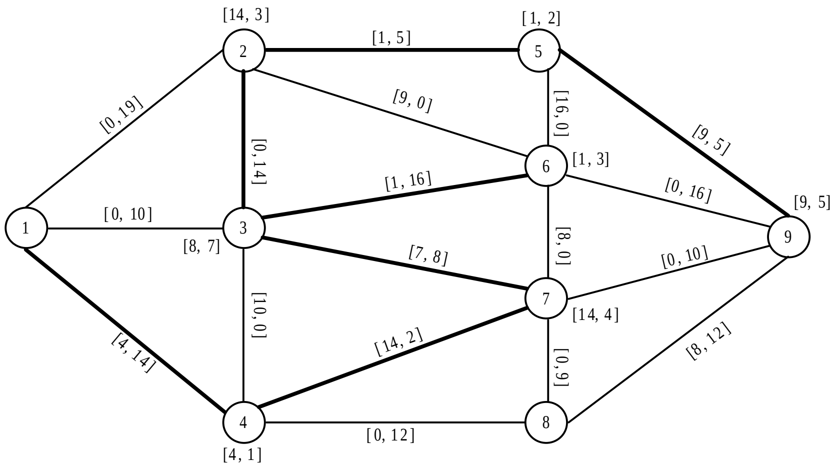

Figure 3. Network with a flow augmenting route (1 – 4 – 7 – 3 – 2 – 5 – 9) and c^{*}= 1

and (1 – 2 – 5 – 9) with lower reliability [0.108, 0.320] and [0.20, 0.49] respectively, having their capacity increased by one to the values of 3 and 6 respectively.

We can simplify it as follows (modification in *Iteration* 4 we denote it as *Iteration* 4^*, and so on.):

Iteration 4^*. The most reliable route H_4 is (1 – 2 – 5 – 9) with interval probability [0.20, 0.49], and minimum capacity $c_4^* = 5 + 1 = 6$, so an interval expected capacity on this route is [1.20, 2.94].

Iteration 6^*. The most reliable route H_6 is (1 – 4 – 7 – 9) with interval probability [0.108, 0.32], and minimum capacity $c_6^* = 2 + 1 = 3$, so an interval expected capacity on this route is [0.324, 0.96].

Iteration 3^*. The most reliable route H_3 is (1 – 2 – 3 – 7 – 9) with interval probability [0.2352, 0.648], and minimum capacity $c_3^* = 8 – 1 = 7$, so an interval expected capacity on this route is [1.6464, 4.536].

Now, we compute the new total maximum flow (*NTMF*), and the new interval expected flow (*NIEF*). Hence,

$$NTMF = c_1^* + c_2^* + c_3^* + c_4^* + c_5^* + c_6^*$$
$$= 10 + 6 + 7 + 6 + 12 + 3 = \mathbf{44};$$

$NIEF$ = [3.36, 7.20] + [1.613, 4.30] + [1.6464, 4.536] + [1.20, 2.94] + [2.16, 5.376] + [0.324, 0.96] = **[10.3034, 25.312]**.

The new interval expected flow (*NIEF*) is greater than the interval expected flow (*IEF*). So, we will set *NIEF* = *IEF* = [10.3034, 25.312].

The maximal flow is now **44**. No new flow augmenting route can be constructed, and from the results of theorem 2, the interval maximum expected flow is **[10.3034, 25.312]**.

The optimal flow in different arcs is given in *table 1* and is shown in *figure 4*.

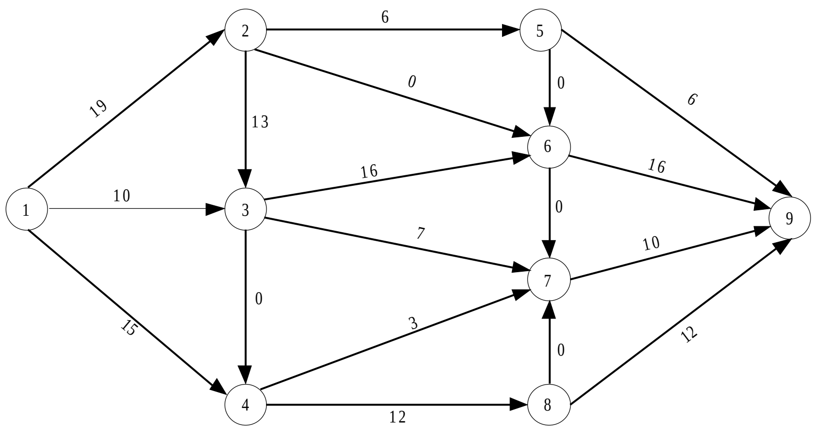

Figure 4. Network with optimal flow

2.2. Numerical Example

Consider the network described in *figure 5*, where an arc (i, j) has two numbers: interval probabilities and capacity. We are again required to find maximum interval expected flow from source node to destination node.

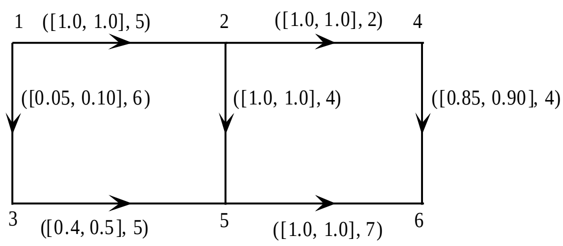

Figure 5. Network with interval probabilities and capacity

Table 1. Optimal flow

Arc	$(\tilde{c}_{ij}, \tilde{c}_{ij}) - (c_{ij}, c_{ji})_7$	Flow	Direct.
1, 2	(19, 0) – (0, 19) = (19, -19)	19	$1 \to 2$
1, 3	(10, 0) – (0, 10) = (10, -10)	10	$1 \to 3$
1, 4	(18, 0) – (3, 15) = (15, -15)	15	$1 \to 4$
2, 3	(14, 0) – (1, 13) = (13, -13)	13	$2 \to 3$
2, 5	(6, 0) – (0, 6) = (6, -6)	6	$2 \to 5$
2, 6	(9, 0) – (9, 0) = (0, 0)	0	-
3, 4	(10, 0) – (10, 0) = (0, 0)	0	-
3, 6	(17, 0) – (1, 16) = (16, -16)	16	$3 \to 6$
3, 7	(15, 0) – (8, 7) = (7, -7)	7	$3 \to 7$
4, 7	(16, 0) – (13, 3) = (3, -3)	3	$4 \to 7$
4, 8	(12, 0) – (0, 12) = (12, -12)	12	$4 \to 8$
5, 6	(16, 0) – (16, 0) = (0, 0)	0	-
5, 9	(14, 0) – (8, 6) = (6, -6)	6	$5 \to 9$
6, 7	(8, 0) – (8, 0) = (0, 0)	0	-
6, 9	(16, 0) – (0, 16) = (16, -16)	16	$6 \to 9$
7, 8	(0, 9) – (0, 9) = (0, 0)	0	-
7, 9	(10, 0) – (0, 10) = (10, -10)	10	$7 \to 9$
8, 9	(20, 0) – (8, 12) = (12, -12)	12	$8 \to 9$

Using the new algorithm, we can find the maximum interval expected flow from source node to destination node.

Iteration 1. First find the most reliable route on the given network using the most reliable route interval algorithm. The most reliable route H_1 is (1 – 2 – 5 – 6), and the reliability of this route is [1.0, 1.0]. The minimum capacity along this route $c_1^* = 4$. The $IEC(H_1)$ on this route is [4.0, 4.0]. Modify the capacity of all arcs on the route H_1, and form a new network.

Iteration 2. The second most reliable route H_2 is (1 – 2 – 4 – 6), and the reliability of this route is [0.85, 0.90]. The minimum capacity along this route $c_2^* = 1$. The $IEC(H_2)$ on this route is [0.85, 0.90]. Reduce the capacity of all arcs on the route H_2, and form a new network.

Iteration 3. The third most reliable route H_3 is (1 – 3 – 5 – 6), and the reliability of this route is [0.02, 0.05]. The minimum capacity along this route $c_3^* = 3$, so the $IEC(H_3)$ on this route is [0.06, 0.15]. We need to modify the capacity of all arcs on the route H_3, and form a new network.

No new route is found in the new network (after iteration 3) with positive capacity. So, we will move to *step* 6.

The total maximum flow (*TMF*), and the interval expected flow (*IEF*) are as follows:

$TMF = c_1^* + c_2^* + c_3^* = 4 + 1 + 3 = \mathbf{8}$;

IEF = [4.0, 4.0] + [0.85, 0.90] + [0.06, 0.15] = **[4.91, 5.05]**.

Iteration 4. Now we will move to *step* 7, and using the „artificial capacities“, we found a flow augmenting route in *figure 6*. It is (1 – 3 – 5 – 2 – 4 – 6), and the *smallest capacity* is 1. Hence, arcs (1 – 3), (3 – 5), (2 – 4) and (4 – 6) have capacity in ($i \to j$) direction (in forward direction), but arcs (2 – 5) is used in ($j \to i$) direction (in backward direction).

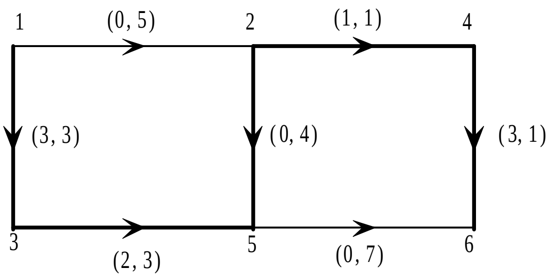

Figure 6. Network with a flow augmenting route (1 – 3 – 5 – 2 – 4 – 6) and $c^* = 1$

We therefore obtain the route (1 – 2 – 5 – 6), with higher reliability [1.0, 1.0], having its capacity reduced by one, and two routes (1 – 2 – 4 – 6) and (1 – 3 – 5 – 6) with lower reliability [0.85, 0.90] and [0.02, 0.05] respectively, having their capacity increased by one.

We can simplify it as follows (modification in *Iteration* 2 we denote it as *Iteration* 2*, and so on.):

Iteration 2*. The second most reliable route H_2 is (1 – 2 – 4 – 6), and the reliability of this route is [0.85, 0.90]. The minimum capacity along this route $c_2^* = 1 + 1 = 2$, so the $IEC(H_2)$ on this route is [1.7, 1.8].

Iteration 3*. The third most reliable route H_3 is (1– 3 – 5 – 6), and the reliability of this route is [0.02, 0.05]. The minimum capacity along this route $c_3^* = 3 + 1 = 4$, so the $IEC(H_3)$ on this route is [0.08, 0.20].

Iteration 1*. The most reliable route H_1 is (1 – 2 – 5 – 6), and the reliability of this route is [1.0, 1.0]. The minimum capacity along this route $c_1^* = 4 - 1 = 3$, so the $IEC(H_1)$ on this route is [3.0, 3.0].

Now, we compute *NTMF*, and *NIEF*. Hence,

$NTMF = c_1^* + c_2^* + c_3^* = 3 + 2 + 4 = \mathbf{9}$;

$NIEF$ = [3.0, 3.0] + [1.7, 1.8] + [0.08, 0.20] = **[4.78, 5.00]**.

The new interval expected flow is less than the interval expected flow.

Hence, from the results of theorem 1, the interval expected flow is **[4.91, 5.05]**, and the maximum flow is **8**.

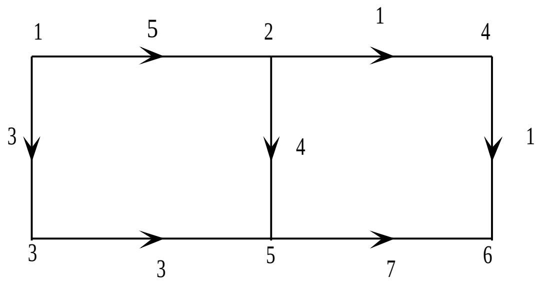

Figure 7. Network with optimal flow

The optimal flow in different arcs is given in *table 2* and is shown in *figure 7*.

Table 2. Optimal flow

Arc	$(\overset{\approx}{c}_{ij}, \overset{\approx}{c}_{ij}) - (c_{ij}, c_{ji})_3$	Flow	Direction
(1, 2)	(5, 0) – (0, 5) = (5, -5)	5	$1 \rightarrow 2$
(1, 3)	(6, 0) – (3, 3) = (3, -3)	3	$1 \rightarrow 3$
(2, 4)	(2, 0) – (1, 1) = (1, -1)	1	$2 \rightarrow 4$
(2, 5)	(4, 0) – (0, 4) = (4, -4)	4	$2 \rightarrow 5$
(3, 5)	(5, 0) – (2, 3) = (3, -3)	3	$3 \rightarrow 5$
(4, 6)	(4, 0) – (3, 1) = (1, -1)	1	$4 \rightarrow 6$
(5, 6)	(7, 0) – (0, 7) = (7, -7)	7	$5 \rightarrow 6$

3. Conclusions

In this paper, a method and algorithm are proposed for solving the maximum flow problem, in a probabilistic setting, when the probabilities of realization of arcs capacities are uncertain. Two numbers are assigned to each (i, j): the interval probabilities, and the capacity. In [13], the author considered the same problem with less degree of uncertainty, and dynamic programming is used to find the most reliable route from source node to destination node.

The most reliable route interval algorithm [5,6,8,10] based on '$\times$' operator is used to find the most reliable route and the reliability of the route. The concept of „artificial capacities" is used to find the flow augmenting route on the network. The flow augmenting route increases the flow from the source node to the destination node. This may increase or decrease or not to change the interval expected flow from the source node to the destination node, since some routes reduce the capacity and other routes increase the capacity. Hence, the sum of all interval expected capacities can be increased or decreased, as well as, the interval expected flow. The applicability of the algorithm is demonstrated by considering two examples.

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