Analytical Methods and Algorithm of Decision LWR- model of Traffic Flow in a Class of Distributed Parameters Systems

Key Words: Analytical solution for the PDE; Green function; distributed parameters systems.

Abstract. The Web service examples depending on the usage of the UDDI are presented. Development of an application, which provides access to various Web Service resource definitions, based on the UDDI registry, is explained and presented. The main aim on this article is through the analytical methods, of partial differential equation P.D.E. resolution, to obtain an exact solution for the PDE presenting the LWR [Lighthill et Whitham, 1955] et [Richards, 1956] traffic flow model and farther to obtain the transfer function of this distributed plant. This model demonstrates the distribution of vehicles on the highs ways and presents the traffic flow as a distributed parameter system D.P.S. The LWR model is a part of the macroscopic traffic flow models based on the fluid mechanics, where the physical model is presenting by a non-linear hyperbolic (or quasi-linear) PDE. The different analytical methods of solution are shown, some of them approximate (wildly used classical methods of solution) and some of them giving an exact solution called the Green function method (integral kernel method). This analytical method of exact solution of the PDE use this Green function as an integral kernel changing the differentials equations in algebraic equations translating the phenomenon of the distribution of vehicles on the high ways. Using this exact method we obtain directly the plant transfer function for the traffic flow model viewed as a DPS. The validation of all obtaining results is made by the comparison analysis with the wildly used numerical solution.

1. Introduction

This paper is devoted to the analytical solution of the *PDE* presenting the macroscopic traffic flow model as a *DPS* where the main aim is to obtain the transfer function of the distributed process. We need this transfer function for future control applications problems as the control of the speed, bottleneck control, ramp metering control etc. This *PDE* take part of the non-linear (or quasi-linear) first order *PDE*, this is actually the *Euler* equation [6, 7] expressing the mass conservation for incompressible fluid, presenting the distribution of vehicles in the time and in the unidirectional space. All the research and obtaining results are important for the further research in this area of traffic flow control, where all the problems concern the correct functionalities of the process and the fluidity of the traffic flow during a period of the time and for a section on the road.

2. Macroscopic Traffic Flow Model LWR

Relations

The LWR macroscopic traffic flow model presents the static model of the fluid mechanic translating the movement of M. Uzunova, D. Jolly, E. Nikolov

a fluid in a pipe. This first order **PDE** of two independent variables (time and unidirectional space) giving us the relation between the density, the flow and the average speed of the traffic flow by the following equations: conservation law (1), the speed relation (2) and the fundamental diagram **F.D.** (3).

(1)
$$\frac{\partial Q(x,t)}{\partial x} + \frac{\partial \rho(x,t)}{\partial t} = 0;$$

(2) $V(x,t) = V_{cs} f(\rho(x,t));$
(3) $q(x,t) = V(x,t) \rho(x,t).$

The **F.D.** (3) is obtaining experimentally and gives all the equilibrium statements, the conductor's behavior and the road situation. There is many proposition of approximation of (3) proposed by **Greenschield** (1934), **Cassidy** (1998), **Greenberg** (1959), **Drake** etc [1-5]. In the further calculation we will use the **Greenschield** approximation model. The most important parameters in **FD** (*figure 1*) are the critical density, maximal density, maximal value of the capacity (flow) and free speed.



Figure 1. Greenschield fundamental diagram

As result of (1)-(3) we have (4)

$$(4) \quad \frac{\partial \rho(x,t)}{\partial t} + \rho(x,t) \frac{\partial V(x,t)}{\partial x} + V(x,t) \frac{\partial \rho(x,t)}{\partial x} = 0$$

(4) is the first order non-linear hyperbolic **PDE** whose analytical resolution is the main aim of this paper. Each solution requires fixing the working point by the initial condition *I.C.* and the boundary conditions *B.C.* (5).

$$u(x, t)_{(x, t:\sigma)} = \rho(x, t)_{(x, t:\sigma)} = f(x) \begin{cases} \neq 0 \\ = 0 \end{cases}$$

(5) $u(0, t)_{(x:\sigma, t)} = \rho(x, t)_{(x:\sigma, t)} = \begin{cases} u_1(t) \Rightarrow q_m \\ 0, BC \\ u(L, t)_{(x=L, t)} = \rho(x, t)_{(x=L, t)} = \begin{cases} u_2(t) \Rightarrow q_{max} \\ 0, BC \end{cases}$

The BC must to be homogenous for the whole space interval. The choice of the IC and BC are very important, because they provide the particular solution of the **PDE**. To continue with the solution using different analytical methods, the non-linearity in the traffic flow model must be approached. This non-linearity is in the velocity relation where the speed/density dependence exists. So to attain the solution it is necessary to make an approximation with the linear one, looking for the assure equilibrium in the model.

Approximation of the non-linearity in the model in dependence with the approximation approach for the speed relations we can briefly show four cases [21-22].

1) Approximation with a constant V(x, t) = c: we consider that the vehicles move with a constant speed value[21-22] (6).

The solution is a set of characteristics. In this case it is a network of line whose slope is the constant speed value (figure 2).





2) Approximation with a space depend function V(x, t) = c(x): we consider that the vehicles move with

information technologies and control a speed changing within along the road (7). We have a function depending only of the space variable for the speed relation.

$$V(x, t) = c(x);$$

$$\frac{\partial \rho(x, t)}{\partial t} + c(x) \frac{\partial \rho(x, t)}{\partial x} = 0;$$

$$\rho(x(s), t(s)) = f(s) = \rho_0 e^{-s} (I.C.);$$

$$\int_{s}^{x} dx' = c(x) \int_{0}^{t} dt' \implies \int_{s}^{s} \frac{1}{c(x)} dx' = \int_{0}^{t} dt' \implies$$

$$log(\frac{x}{s}) = ct \implies s = x e^{-ct}$$

$$\rho(x, t) = x e^{-ct}$$

In this case the characteristics solution is a translation of the initial condition (in the solution e.g. if it is an exponential function) distorted within the space dependent speed.

3) Approximation with a time depend function

V(x, t) = c(t): we consider that the vehicles move with a speed changing within the time interval (8) [21-22].

$$V(x,t) = c(t);$$

$$\frac{\partial \rho(x,t)}{\partial t} + c(t) \frac{\partial \rho(x,t)}{\partial x} = 0;$$
(8)
$$\rho(x(s), t(s)) = f(s) = \rho_0 e^{-s} (I.C.);$$

$$\int_{s}^{x} dx' = \int_{0}^{t} c(t) dt' \Rightarrow x - s = log(ct) \Rightarrow$$

$$s = (x - log ct) \Rightarrow \rho(x,t) = f(x - log ct).$$

Here as a solution a translation of the initial condition is obtained, but with a non-constant speed.

4) Approximation with a function depending as
follows:
$$V(x,t) = c(\rho), (9) [21-22]:$$
$$V(x,t) = c(\rho) \Rightarrow \frac{\partial \rho(x,t)}{\partial t} + c(\rho) \frac{\partial \rho(x,t)}{\partial x} = 0;$$
$$(9) \begin{cases} x(s) = s \\ t(s) = 0 \end{cases} \Rightarrow \rho(s,0) = f(s) = \rho(x,t);$$
$$\int_{s}^{s} dx' = \int_{0}^{t} c(\rho(x,t)) dt' - solution : x - s = f(s) t;$$
$$\rho(x,t) = f(x - \rho(x,t) t).$$

The conservation of the speed along the characteristics solution of our **PDE** is obtained. So the result in the nonlinear case is a characteristics solution of the **PDE** using the method of lines whose slope depends on the initial conditions and which are propagated along the space with a constant speed. For trapezoidal view of vehicles distribution (trapezoidal input signal for the system) the result is the time/space distribution as shown on (*figure 3*).





3. Aproximate and Exact Analytical Methods for Solution of PDE Describing the LWR Traffic Flow Model

Approximate Solutions – Classical Analytical Solving Methods

There are some analytical methods for obtaining the solution of one **PDE**, where the non-linearity is already approximated. *Figure 4* represents a scheme of different resolution methods of **PDE**'s – analytical and numerical as well. But in the following research only the analytical methods will be treated and a comparison and validation with the numerical ones will be made.



Figure 4. PDE solving methods

Exact Solution - Green Function Method

This is an exact method for solving linear, quasi-linear, non-linear and all the different **ODE** and **PDE** with a boundary condition [13-16], giving the distributed nuance of the problem, where the **Green** function is used as a integral kernel for solving the non-linear **PDE** of the **LWR** model [7, 11], [13-16]. The main idea is based on the distribution theory of the potential source, where the distribution of the vehicles on the high ways can be replaced by a concentric source with a specific boundary conditions. We look for a solution of the **PDE** presenting our distributed parameter process for obtaining the transfer function and applying it in the automation system using different control strategies, which depend on the road network and the real situation on it. The basic theory gives the next condition: if a

solution exists in one point of the space, we can write the equation as a convolution product of a function $\varphi(x)$ and a differential operator **D** equal to another function called source j(x) (10). Actually, we replace the function in the **PDE** by the convolution product of differential operator and the searching **Green** function.

$$D \varphi (x) = j (x)$$

$$D = \sum_{\alpha=0}^{m} a_{\alpha} p^{\alpha}$$

$$\varphi (x) = (G^* j) (x) = \int_{-\infty}^{\infty} G (x - \xi) j (\xi) d\xi$$

In this way this function called **Green** can be found under the condition that the point source is a **Dirac** delta function in the space and in the time as is shown (11). In case of homogeneous **PDE** a distributed source should be applied and then the non-homogeneous **PDE** solved decomposing the distributed source by a set of local concentrated sources.

That means that a defined number of under space intervals is obtained and the replacement of the initial condition (our searching function) in each of them is searched. That could be possibly viewed in the theory of distribution (figure 5) using the **Dirac Delta functions** providing this property of translation. If on the input point where the time and the space have an initial zero values and there is a group (constant number) of vehicles they will appear without losses in the next segment after a defined interval of time.

11)
$$EDP$$
 :

$$\begin{cases} DG(x,t) = \delta(x) \delta(t) \\ DG(x-y, t-\tau) = \delta(x-y) \delta(t-\tau) \end{cases}$$



Figure 5. Distribution effect of the initial condition in the time and for each under-interval e.g. step function translating constant number of vehicles on the high way

Moreover there exist two ways to solve the **PDE** by the **Green** function [10], [18], based on the inverse **Laplace** (**Fourier**) transform **I.L(F).T**. and on the **Hadamard** variation formula. Thereafter, for the **LWR** model the first method with zero initial and homogenous boundary conditions will be used.

4. Numerical Example Using the *Green* Function Method and Algorithm Through the IL(F)T

1) For the LWR Model we Have (12) [1]-[6]

(12)
$$\frac{\partial \rho(x,t)}{\partial t} + V(x,t) \frac{\partial \rho(x,t)}{\partial x} + \frac{\partial V(x,t)}{\partial x} \rho(x,t) = 0;$$
$$\frac{\partial \rho(x,t)}{\partial t} + V_f \left(1 - \frac{\rho(x,t)}{\rho_{\text{max}}}\right) \frac{\partial \rho(x,t)}{\partial x} + \frac{\partial V_f \left(1 - \frac{\rho(x,t)}{\rho_{\text{max}}}\right)}{\partial x} \rho(x,t) = 0.$$

For the future calculations the constant value of the speed $V(\rho) = 0.5 V_f m/s$ is used (13).

$$\left(\frac{\partial}{\partial t} + 0.5 V_j \frac{\partial}{\partial x}\right) \rho(x,t) = j(x,t), j(x,t) = 0;$$

$$\begin{array}{l} 13) \quad D \ \mathcal{P}\left(x,t\right) = f\left(x,t\right), f\left(x,t\right) = 0;\\ DG\left(x,\,\xi,\,t,\tau\right) = \delta\left(x-\xi\right)\,\delta\left(t-\tau\right);\\ \left(\frac{\partial}{\partial t} + 0.5 V_f \frac{\partial}{\partial x}\right) G\left(x,\xi,t,\tau\right) = \delta\left(x-\xi\right)\delta\left(t-\tau\right) \end{array}$$

The IC and the BC are as follow (14):

(14)
$$\begin{cases} \rho(x,0) = \rho_0(x) = 0, & t \ge 0; \\ \rho(0,t) = \rho(L,t) = \rho(t) = 0, & 0 \le x \le L. \end{cases}$$

2) The Green Function Solution (13) [10-16],[19]

The **Green** function presents the inverse of the operator and the source convolution. Thus, moreover one can say that the convolution product presents the **Fourier** transform of the function. So, an operational form of this **PDE** should be obtained. The **Laplace** transform is applied and the following (15) is got:

(15)
$$\left(p+0.5 \ V_f \frac{d}{dx}\right) G(x, x_0, p) = \delta(x-x_0) \ e^{-p\tau} + \rho_0(x, 0);$$

I.C.: $\rho_0(x, 0) = 0.$

After the normalization (16) is valid

(16)
$$G(x, x_0, p) e^{p\tau} = \hat{G}(x, x_0, p);$$
$$(p+0.5 V_f \frac{d}{dx}) \hat{G}(x, x_0, p) = \delta(x-x_0).$$

The *Fourier* transform and the separation of the *Green* function and the operator are shown on (17):

$\begin{pmatrix} p \ \hat{G}(f, x_0, p) \end{pmatrix} + \begin{pmatrix} 0.5 \ V_f(i 2 \pi f) \ \hat{G}(f, x_0, p) \end{pmatrix} = e^{-i2\pi f x_0};$ $(17) \ \hat{G}(f, x_0, p) = \frac{e^{-i2\pi f x_0}}{(p + 0.5 V_f(i 2 \pi f))};$ $G(f, x_0, p) = \frac{e^{-p\pi} e^{-i2\pi f x_0}}{0.5 \ V_f} \left(\frac{1}{0.5 \ V_f} p + (i 2 \pi f)\right)^{-1}.$

To obtain the final solution of the **PDE** using the **Green** function through the integral kernel an **IFT** has to be made(18).

(18)

$$\hat{G}(x, x_0, p) = \frac{1}{0.5 V_f} u(x-x_0) e^{-\frac{p(x-x_0)}{0.5 V_f}};$$

$$u(x-x_0) = 1.$$

The final solution is given by (19)

(19)
$$G(x, x_0, p) = \frac{1}{0.5 V_f} (x - x_0) e^{-\frac{p\left(x - x_0 - \frac{\tau}{p}\right)}{0.5 V_f}}.$$

The general form of the solution is shown on (20). The obtained result represents a delay transfer function, depending of the time and the space as well. This time/space delay function shows that there are no loses in the model, analogically that there is a constant value vehicles speed and that the same packet of vehicles has to be found within the next segment but after defined time τ . After equivalent transformations the classical form of transfer function (20) is reached:

$$G(x, x_{0}, p) = W(x, x_{0}, p) = \frac{k_{LWR} e^{-p\tau_{LWR}}}{T_{LWR} p + 1},$$
(20) $(x - x_{0}) = L; V(x, t) = V_{f} (1 - \rho(x, t) \rho_{max}^{-1});$
 $k_{LWR} = f(L, \rho, V_{f}); T_{LWR} = \tau_{LWR} = f(L, \rho, V_{f}).$

To complete the research the obtained analytical result (20) of traffic function is to be used to construct an automation system of *LWR* model. In the simulation example the density, the speed and the length variation are shown and the results are on (*figure 5*) and (*figure 6*). For the example the following initials values are (21):

$$\frac{\rho(x,t)}{\rho_{\max}} = [0.1, 0.2], [-];$$
(21)

$$L = x - x_0 = 1 [km];$$

$$V_t = [80, 100, 120, 140, 160], [km/h].$$

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Figure 6. Time responses characteristics for the obtained Green solution of LWR traffic flow model



Figure 7 a. Frequency responses: Nyquist 2D - characteristics for the obtained Green solution of LWR traffic flow model



Figure 7 b. Frequency responses: *Nyquist 3D* -characteristics for the obtained *Green* solution of *LWR* traffic flow model



Figure 7 c. Frequency responses: Nyquist - characteristics for the real corposants for the obtained Green solution of LWR model



Figure 7 c. Frequency responses: *Nyquist* -characteristics for the imaginary corposants for the obtained *Green* solution of *LWR* model



Figure 8. Frequency responses: *Bode* - characteristics for the real and the imaginary corposants for the obtained *Green* solution of *LWR* model

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Figure 9.a. Frequency resp.: *Nichols*-characteristics for the real and the imaginary corposants for the obtained *Green* solution of *LWR* model

5. Simulation Results of the Obtained Transfer Function Presenting the *LWR* Traffic Flow Model as a DPS

Some important point of this research is to confirm the obtained results. The analytical solution precision has been proved using other, wildly known analytical solution methods, but it was more interesting to prove it by simulation of the **DPS** transfer function in an automation system.

For the aims of the simulation research the initials conditions (21) are set and afterwards the approximation is implied which leads (19) to (20). The synthesis of the simulation gives the time (*figure 6*) and the frequency responses – **Nyquist** (*figure 7a.*), **Bode** (*figure 8*) and **Black-Nichols** (*figure 9a*) and 3D (*figure 9b.*).

From the simulations results one can conclude that the traffic flow model represents one **DPS** where the density of the vehicles on the road is shown by the step response characteristics and by the frequencies as well. The different simulated models give the information about influence if the speed variation and for 2 different points of the **FD** for the same length of road section. One can conclude that the model is correct because of the stabilities performances observed on all the characteristics (*figure 5*)-(*figure 8*).

The hypothesis of this different constant value of the speed means that the vehicles move on the road without loses. That corresponds to a delay transfer function of the **LWR** macroscopic traffic flow model plant as was obtained by the analytical solution as well.

Conclusion

This paper proposes an extended study of the analytical method and algorithm of solution of the **PDE** where the plant is the macroscopic traffic flow model wielding as a time and space **DPS**. Because of the property of this distributed plant model (the fact that the distribution of vehicles on the high ways is a distributed system) and the type of the equation which is a non-





linear **PDE** to obtain an exact solution a specific mathematical method called **Green** function and the distribution theory are used.

Farther, a different approximation of the non-linearity was revealed. An important step for obtaining the exact solution is the choice of the *I.C.* and *B.C.* fixing the working point. Using this method the transfer function of the *DPS* of the *LWR* traffic flow model is obtained. For validating all the results a simulation analysis was proposed [18-21].

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