Polynomial Algorithms for Solving the Most Reliable Route Problem

A. Hossain, G. Gatev

Key Words: *Networks; possibility; interval possibility; most reliable route; interval most reliable route.*

Abstract. *The uncertainty about the reliability of a route is represented in a possibilistic setting. The concept of interval possibility is introduced, as a generalization of fuzzy set concept of possibility, to deal with a higher degree of uncertainty. Four simple algorithms are proposed for solving the Most Reliable Route Problem under parametric uncertainty. The aim is to find the most reliable route on a network that maximizes the possibility of not being stopped on the route. The possibilities on the route segments are uncertain. The applicability of the results is demonstrated by considering several examples.*

1. Introduction

We use different types of networks in our everyday lives, for example, electrical networks, telephone networks, transportation networks (i.e., highway, rail, etc.), natural gas supply networks, etc. The origins of network analysis are very old. The network models, methods, and algorithms are extensively studied during the last fifty years and described in [1,2,3,9,10,11, 17,18,19,20,21].

Consider a connected network G, $G = (N, A)$, where N is the set of all nodes and A is the set of arcs. The cardinality of N and A are denoted by $|N|$ and $|A|$ respectively, and $|N| = n$, $|A| = m$.

There are many reasons why network models and algorithms are widely used, for instance, they exactly represent the real world systems, they facilitate extremely efficient solution to large real problems, they can solve problems with significantly more variables and constraints than can be solved by other optimization techniques, etc. [17].

One of the earliest and computationally most efficient algorithm for cyclic network, that is, for a network that contains loops, is given in [2]. The earliest review of shortest route problem is given in [18].

A method is proposed to find the most reliable route in a given network in [19,20]. The probability of an arc is certain. The author converted probability to log probability. Then the shortest route algorithm is used to find the shortest distance (log). Finally, this log probability is converted back to non-log probability.

In [2], the author assumes n nodes, and the existence of at least one route between any two nodes. Two fundamental problems are considered: to obtain the tree of minimum total length between the n nodes, and to find the route of minimum total length between two given nodes.

In [1], the authors propose an insightful alternative method for shortest route problems, which reduce the upper bound of running time, and make empirical comparisons for a certain class of networks. Reaching, Pruning, and Buckets are the three concepts that are used in these methods. Reaching is a label setting scheme, reaching allows a network to be pruned during computation of some of its nodes and/or branches, and bucket is a list of nodes whose labels fall within a given range.

New polynomially bounded shortest route algorithms, called the Partitioning Shortest Route algorithm (PSR algorithm), to find the shortest route from one node to all other nodes in a network are given in [9]. The authors discuss six variants of the PSR algorithms: an algorithm for negative arc lengths, but no negative cycles, and two algorithms for nonnegative arc lengths, augment the PSR algorithm to maintain a property sharp by Shier and Witzgall, and the other three variants augment the PSR algorithm to maintain a property called near-sharp for nonnegative arc lengths.

In [13], a new label correcting algorithm based on the use of buckets of queues is given. A graph reduction technique for real road networks is also presented, and this reduction technique decreases the number of nodes and arcs in the network.

An insightful interval algorithm is proposed for solving network problems under parametric uncertainty in [5]. The exact values of the parameters of a given network are unknown, but upper and lower limits within which the values are expected to fall are considered. The interval algorithm is developed on the base of midpoint and half-width representation of intervals. Considerable unification and simplification are obtained by using the mean-value lemma. This interval algorithm is applicable when the parameters of a given network are interval and non-interval.

In [18], the authors gave the formulation of the shortest route problem through a network: A set of N cities (nodes) is given with every two connected by a road (link). The distances are given between cities, and the distance from city i to city j is not always equal to the distance from city j to city i. All distances are assumed to be non-negative. The aim is to find a route from any one city to any other city that minimizes the total distance. If there is no link between two cities, the distance is considered infinite.

In many practical cases, the parameters of the network models are not exactly known, they are uncertain. A typical way to express these uncertainties in the edge weights is to utilize tools based on probability theory, interval analysis, fuzzy sets theory.

The interval arithmetic is used as a tool to solve many problems which are difficult to deal by classical solution technique [5,6,7,8,14].

After the seminal work of Zadeh [22], many authors have discussed fuzzy logic as a tool to deal with uncertainty, see, e.g. [4,12,23].

The key concept of possibility, its close connection with the concept of membership in a fuzzy set, and its important role in the representation of meaning in the management of uncertainty and in application of the fuzzy approach to decision analysis, have been developed and treated in [4,23].

The aim of this paper is to develop polynomial algorithms for solving the Most Reliable Route Problem under parametric uncertainty. The most reliable route problem is formulated in a probabilistic setting and solved using a shortest route algorithm [19,20].

In this paper, we consider higher degree of uncertainty. First, we use *possibilities* to represent the plausibility of not being stopped on the route. Thereafter, the degree of uncertainty is further increased by introducing the concept of *interval possibility* as an extension of fuzzy set concept of possibility [4]. Interval possibilities are more appropriate in the case, when the value of possibilities are uncertain but expected to fall within given intervals.

The paper is organized as follows. The interval analysis concepts and some fuzzy graphs concepts are discussed in Theoretical Preliminaries in the second section. Polynomial algorithms for Most Reliable Route Problem, as well as numerical examples to illustrate the applicability of the algorithms are presented in the third section. The obtained results are discussed in the conclusion in section 4.

2. Theoretical Preliminaries

First the interval analysis concepts are introduced [14,15,16].

Let R be an interval. We will denote its lower (left) endpoint by $\underline{r}$ and its upper (right) endpoint by $\overline{r}$, so that $R=[\underline{r},\overline{r}]$.

The set of all intervals will be denoted by $I(R)$. Let $R, S \in I(R)$, and let * denote any of the interval arithmetic operations, * = +, -, x, /. Then the set theory definition of the interval arithmetic operations is as follows:

(1) $R*S=\{r*s \mid r\in R, s\in S\}$

It follows that the sum of $R=[\underline{r},\overline{r}]$, $S=[\underline{s},\overline{s}]$ denoted by $R+S$, is the interval

$$R+S=[\underline{r},\overline{r}]+[\underline{s},\overline{s}]=[\underline{r}+\underline{s},\overline{r}+\overline{s}]$$

The product $R\times S$ is again an interval

$$R\times S=[\min\{\underline{r}\underline{s},\underline{r}\overline{s},\overline{r}\underline{s},\overline{r}\overline{s}\}, \max\{\underline{r}\underline{s},\underline{r}\overline{s},\overline{r}\underline{s},\overline{r}\overline{s}\}]$$

For $R,S>0$ the definition reduces to

(2) $R\times S=[\underline{r}\underline{s},\overline{r}\overline{s}]$

The half-width of an interval $R=[\underline{r},\overline{r}]$ is the real number, $w(R)=\frac{1}{2}(\overline{r}-\underline{r})$, and the midpoint of R is the real number, $m(R)=(\underline{r}+\overline{r})/2$.

Using the set inclusion relation $\subseteq$ and the relation $\leq$, we can define the supremum-like and infimum-like intervals:

(3) $\sup(R,S)=[\sup(\underline{r},\underline{s}),\sup(\overline{r},\overline{s})]$

(4) $\inf(R,S)=[\inf(\underline{r},\underline{s}),\inf(\overline{r},\overline{s})]$

To compare intervals the concept of metric ρ is introduced. For each R and S in $I(R)$ the distance ρ is defined by

(5) $\rho(R,S)=\frac{1}{2}\{|\underline{r}-\underline{s}|+|\overline{r}-\overline{s}|\}$

Now the intervals R and S can be compared. The following important results hold [5].

$R\leq S$ if and only if

(6) $\rho(R,\inf(R,S))\leq\rho(S,\inf(R,S))$

In a similar way,

$R\geq S$ if and only if

(7) $\rho(R,\sup(R,S))\leq\rho(S,\sup(R,S))$

Two intervals R and S are said to be equivalent $R \sim S$ if the following condition holds:

(8) $\rho(R, \sup(R, S)) = \rho(S, \sup(R, S))$

(9) $\rho(R, \inf(R, S)) = \rho(S, \inf(R, S))$

It means that $|\underline{r} - \underline{s}| = |\overline{s} - \overline{r}|$, i.e., the midpoints of R and S coincide.

In practical cases when $R \sim S$ and one have to make a choice in the sense of ≤, the condition (6) should be modified. We say that $R \leq S$ if

(10) $\rho(R, \inf(R, S)) = \rho(S, \inf(R, S))$ and $\underline{r} \leq \underline{s}$

or

(11) $\rho(R, \inf(R, S)) = \rho(S, \inf(R, S))$ and $\overline{r} \leq \overline{s}$.

We use, further, the notation $R \leq S$ in the usual sense, when $\underline{r} \leq \underline{s}$ and $\overline{r} \leq \overline{s}$, and in the case of inclusion, $R \subseteq S$, when $\rho(R, \inf(R, S)) \leq \rho(S, \inf(R, S))$.

The conditions (6) and (7) lead to the following result, as proven in [5].

Let $m(P)$ denote the midpoint of P, $m(P) = (\underline{p} + \overline{p})/2$. Then

(12) $R \leq S$ if and only if $m(R) \leq m(S)$

Let $[m(R), \Delta(R)]$ denote the interval R, $R = [\underline{r}, \overline{r}]$, where $m(R) = (\underline{r} + \overline{r}) / 2$ is the midpoint of R, and $\Delta(R) = (\underline{r} - \overline{r}) / 2$ is the half-width of R, so that

$R = [m(R) - \Delta(R), m(R) + \Delta(R)]$,

or, using the new notation

(13) $R = [m(R), \Delta(R)]$

The following result is easily shown:

Let R, S, and $T \in I(R)$. Then $T = R + S$ if and only if

(14) $m(T) = m(R) + m(S)$

(15) $\Delta(T) = \Delta(R) + \Delta(S)$

Now we introduce some fuzzy graphs concepts [12].

We shall consider in the finite graph G, $G \subset E \times E$ a path from x_{i1} to x_{ir}, that is an ordered r-tuple $P = (x_{i1}, x_{i2}, ..., x_{ir})$ where $x_{ik} \in E$, $k = 1,2,...,r$ and with the condition

$\forall (x_{ik}, x_{ik+1})$: $\mu_{\Re}(x_{i_k}, x_{ik+1}) > 0$, $k = \overline{1, r-1}$

Let $X \wedge Y$ denote the operator min (X, Y). With each route a value is associated by

$$(16) \quad \begin{aligned} & l(x_{i_1},...,x_{i_r}) = \\ & \mu_{\Re}(x_{i_1}, x_{i_2}) \wedge \mu_{\Re}(x_{i_2}, x_{i_3}) \wedge ,..., \wedge \mu_{\Re}(x_{i_{r-1}}, x_{i_r}) \end{aligned}$$

Let $H(x_i;x_j)$ be the set of all routes between x_i and x_j.

$$(17) \quad \begin{aligned} & H(x_i; x_j) = \\ & \{h(x_i; x_j) = (x_{i_1} = x_i, x_{i_2}, ., x_{i_r} = x_j) \mid \\ & x_{i_k} \in E, k = \overline{2, r-1}\} \end{aligned}$$

The strongest route $H^*(x_i,x_j)$ from x_i to x_j can be obtained

$$(18) \quad \begin{aligned} & l^*(x_i, x_j) = \\ & \bigvee_{H(x_i,x_j)} l(x_{i_1} = x_i, x_{i_2},..., x_{i_{r-1}}, x_{i_r} = x_j) \end{aligned}$$

where $X \vee Y = \max \{X, Y\}$.

The value defined by (16) may be extended to operators other than '$\wedge$' under the restriction that these considered have the properties of associativity and monotonicity. Such an operator is for example, the product operator '$\times$' (ordinary multiplication), for which

If $a,b \in [0, 1]$, than $a \times b \le a \wedge b$

3. Polynomial Algorithms for Most Reliable Route Problem

The aim is to develop simple algorithms for solving the most reliable route problem, when the possibilities of not being stopped on the segments of the route are uncertain. The concept of interval possibility is introduced as an extension of the fuzzy set concept of possibility to describe the uncertainty that usually exists when possibilities have to be evaluated.

The algorithms are based on the strongest route concept given by (18). Let π_{ij} denote the interval possibility of not being stopped on the arc (i, j). Following [4], we consider the possibility as the degree of truth or the plausibility of an assertion, in the case, the plausibility of not being stopped on the arc (i, j).

3.1. Most Reliable Algorithm Based on '$\wedge$' Operator for Acyclic Network

Let P_j denote the maximum possibility from node 1 to node j. By definition $P_1 = 1.0$. The values of P_j, $j = \overline{2, t}$ are computed recursively using the formula

$$(19) \quad P_j = \max_{i \in N_j} \min\{P_i, \ \pi_{ij}\}$$

where i ranges over the set N_j of all immediately preceding nodes linked by an arc to the node j, and π_{ij} denotes the possibility between the current node j and its predecessor node i.

To obtain the optimal solution of the strongest route problem, it is necessary to identify the nodes encountered along the route. The following labeling of the node j is used:

(20) *Node j Label* $= [P_j, k]$

where k is the node immediately preceding the node j, which yields P_j.

Consider the situation when there exist several equivalent routes to a given node. For example, in *figure 1*, there are three equivalent routes $(1 - i - r)$, $(1 - j - r)$ and $(1 - z - r)$ with the same possibility of 0.5. A possible way to deal with the situation is to consider three equivalent strongest route solutions. On the other hand, it is easily seen that the route $(1 - z - r)$ is preferable to the two remaining routes. As a tool to find a unique preferable route we can use the following procedure:

$$(21) \quad s^* = \arg \max_{s=i,j,z} \{P_s + \pi_{sr}\}$$

that is, we choose as immediately preceding node the node s^* which yields the greatest value of the sum $(P_s + \pi_{sr})$. In the example on *figure 1*, the label of node r becomes $[0.5, z]$. Adding possibilities is meaningless. Hence, the procedure (21) is just a „tool" to select a preferable route, and to try to break tie in a non-arbitrary manner.

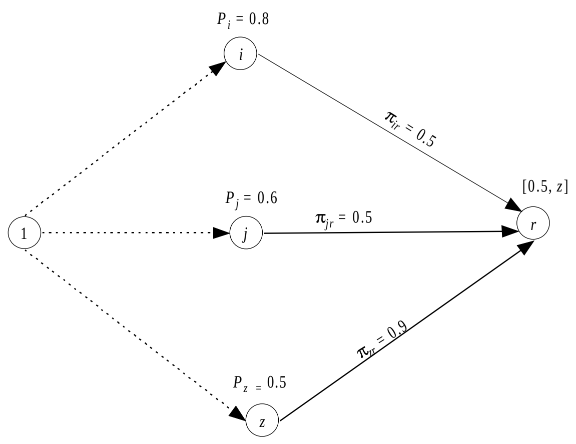

Figure 1

We are considering an acyclic network with natural consecutive numbering of nodes from 1 to n, such that the number of any node j is greater than the number of all immediately preceding nodes linked by arcs to node j. The problem is to find the most reliable route from the starting node 1 to a destination node t, $t \le n$.

The algorithm based on '$\wedge$' operator consists of the following generalized steps:

***Step* 1**: Set j = 1. Assign to the source node (node 1) the label [1.0, –].

***Step* 2**: Set $j = j + 1$. Compute the possibility P_j to node j using the formula (19). Label the node j by using the labeling procedure (20).

If there are several equivalent routes to node j, use (21) to determine the preferable preceding node s^*.

Break a remaining tie arbitrarily, if any.

***Step* 3**: If $j = t$ go to *step* 4, else go to *step* 2.

***Step* 4**: Obtain the optimum route H between nodes 1 and t by tracing backward from node t through the nodes using label's information.

3.1.1. Analysis of the Complexity of the Most Reliable Route Algorithm Based on '$\wedge$' Operator for Acyclic Network

Consider the network in *figure 2*. The cardinality $|N_j|$ of the set of entering arcs N_j into node j is (j – 1), $|N_j| = j - 1$.

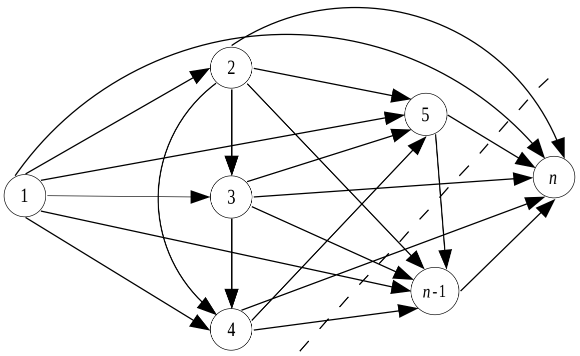

Figure 2

$P_2 = \min\{P_1, \pi_{12}\} \Rightarrow$ We need only (1 + 0) = **1** comparison to determine P_2.

$P_3 = \max\{\min(P_1, \pi_{13}),\ \min(P_2, \pi_{23})\} \Rightarrow$ We need only (2 + 1) = **3** comparisons to determine P_3.

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$P_{(n-1)} = \max\{\min(P_1, \pi_{1(n-1)}), \min(P_2, \pi_{2(n-1)}), \min(P_3, \pi_{3(n-1)}), \min(P_4, \pi_{4(n-1)}), \min(P_5, \pi_{5(n-1)})\} \Rightarrow$ We need only **((*n* – 2) + (*n* – 3))** comparisons to determine $P_{(n-1)}$.

$P_n = \max\{\min(P_1, \pi_{1n}), \min(P_2, \pi_{2n}), \min(P_3, \pi_{3n}), \min(P_4, \pi_{4n}), \min(P_5, \pi_{5n}), ..., \min(P_{n-1}, \pi_{(n-1)n})\} \Rightarrow$ We need only **((*n* – 1) + (*n* – 2))** comparisons to determine P_n.

Hence, to obtain P_j we need (j – 1) + (j – 2) = (2j – 3) comparisons and $j = \overline{2, n}$.

The total number of comparisons is $\sum_{j=2}^{n}(j-1) + \sum_{j=2}^{n}(j-2)$.

We set $\chi = j - 1$ and $\delta = j - 2$ and we obtain

$$\sum_{\chi=1}^{n-1}\chi + \sum_{\delta=1}^{n-2}\delta =$$

$$= \frac{(n-1)\times n}{2} + \frac{(n-2)\times(n-1)}{2}$$

$$= \frac{(n-1)(n+n-2)}{2} = (n-1)^2$$

So, it is a *polynomial* algorithm, with complexity $O(n^2)$.

Numerical example

Consider the network in *figure 3*, with possibilities π_{ij} given along the arcs. Determine the most reliable route from node 1 to node 9.

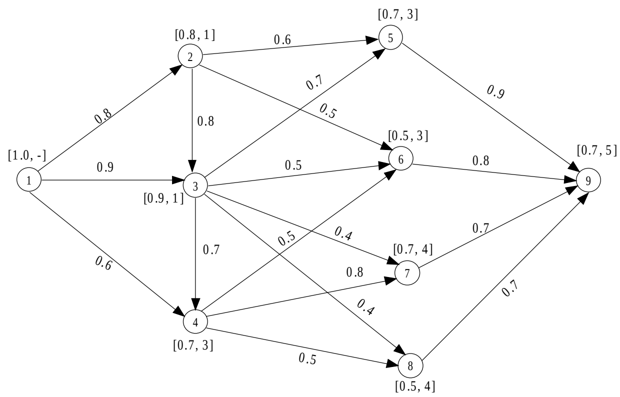

Figure 3

Using the algorithm as we have described in section 3.1 we obtain the following results:

P_1 = [1.0, -];

$P_2 = \min\{P_1, \pi_{12}\}$, = min {1.0, 0.8} = = 0.8 with label [0.8, 1];

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$P_6 = \max_{i=2,3,4} \min\{P_i, \pi_{i6}\} =$

= max {0.5, 0.5, 0.5} = 0.5, with labels [0.5, 2], [0.5, 3], [0.5, 4]

From (21) it follows

s^*= arg max {(0.8 + 0.5), (0.9 + 0.5), (0.7 + 0.5)} = 3;

So, P_6 = 0.5, with label [0.5, 3];

and so on. The computational results are summarized in *figure 3*.

After having P_9, our problem is almost solved. Note that we have found the most reliable routes from node 1 to any node in the network. The optimal solution is obtained starting from node 9 and tracing backward using the label information:

9 ⇒ (0.7, 5) ⇒ 5 ⇒ (0.7, 3) ⇒ 3 ⇒ (0.9, 1) ⇒ 1

So, the most reliable route H is **1 → 3 → 5 →9**, with possibility **0.7**.

The complexity is bounded by $O((n-1)^2)$=64 comparisons.

3.2. Most Reliable Route Algorithm Based on '×' Operator for Acyclic Network

Consider an acyclic network with natural consecutive numbering of nodes from 1 to n, such that the number of any node j is greater than the number of all immediately preceding nodes linked by arcs to node j. The task of this new algorithm is to find the most reliable route from the starting node 1 to a destination node t, $t \le n$.

The parameters P_j and π_{ij}, and the labeling of node j, namely $[P_j, k]$ have the same meaning as in the algorithm based on '$\wedge$' operator in section 3.1. The only difference is that the maximum possibility P_j from node 1 to node j is obtained using the recursive formula

(22) $P_j = \max_{i \in N_j}\{P_i \times \pi_{ij}\}$

where i ranges over the set of all preceding nodes N_j.

The most reliable route algorithm based on '$\times$' operator consists of the following generalized steps:

***Step* 1**: Set j = 1. Assign to the source node (node 1) the label [1.0, -].

***Step* 2**: Set $j = j + 1$. Compute the possibility P_j to node j using the formula (22). Label the node j by using the labeling procedure (20).

Break a remaining tie arbitrarily, if any.

***Step* 3**: If $j = t$ go to *step* 4, else go to *step* 2.

***Step* 4**: Obtain the optimum route H between nodes 1 and t by tracing backward from node t through the nodes using label's information.

3.2.1. Analysis of the Complexity of the Most Reliable Route Algorithm Based on '×' Operator for Acyclic Network

Consider the network in *figure 2* (see, section 3.1.1). The cardinality $|N_j|$ of the set of entering arcs N_j into node j is (j – 1), $|N_j| = j - 1$.

$P_2 = \{P_1 \times \pi_{12}\}$ ⇒ We need only **1** multiplication to determine P_2.

$P_3 = \max\{(P_1 \times \pi_{13}),(P_2 \times \pi_{23})\}$ ⇒ We need only **2** multiplications and only **1** comparison to determine P_3.

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$P_{n-1} = \max\{(P_1 \times \pi_{1(n-1)}),(P_2 \times \pi_{2(n-1)}), (P_3 \times \pi_{3(n-1)}), ..., (P_{(n-2)} \times \pi_{(n-2)(n-1)})\}$ ⇒ We need only **(n – 2)** multiplications and **(n – 3)** comparisons to determine P_{n-1}.

$P_n = \max\{(P_1 \times \pi_{1n}),(P_2 \times \pi_{2n}), (P_3 \times \pi_{3n}), (P_4 \times \pi_{4n}), (P_5 \times \pi_{5n}), ...,(P_{(n-1)} \times \pi_{(n-1)n})\}$ ⇒ We need only **(n – 1)** multiplications (*multi*) and **(n – 2)** comparisons (*comp*) to determine P_n.

Hence, to obtain P_j we need only (j – 1) multiplications and (j – 2) comparisons.

The total number of multiplications is $\sum_{j=2}^{n}(j-1)$.

The total number of comparisons is $\sum_{j=2}^{n}(j-2)$.

We set $\chi = j - 1$ and $\delta = j - 2$, and we obtain:

The total number of multiplications is ϕ,

$$\phi = \sum_{\chi=1}^{n-1}\chi = \frac{(n-1)\times n}{2}$$

The total number of comparisons is φ,

$$\varphi = \sum_{\delta=1}^{n-2}\delta = \frac{(n-2)\times(n-1)}{2}$$

The running time of the algorithm is limited by $O(multi = \phi, comp = \varphi)$.

Numerical Example

We will consider the network given in *figure 4* to illustrate the algorithm. The problem is to determine the most reliable route from node 1 to node 9. The computational results are given in the same figure.

The computation results are as follows:

$P_1 = [1.0,-]$;

$P_2 = \{P_1 \times \pi_{12}\} = \{1.0 \times 0.8\}$ = 0.8, with label [0.8, 1]; and so on.

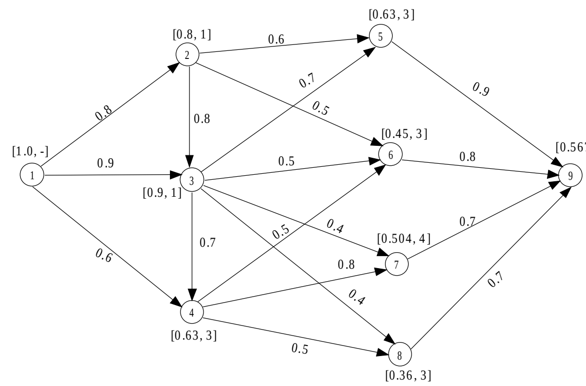

Figure 4

The optimal solution is obtained tracing backward from node 9 and using the label information **1 → 3 → 5 → 9**. The corresponding possibility is **0.567**.

Comment

Equivalent routes may appear more often when the first algorithm is used. In many cases the second algorithm which utilizes the operator '×' automatically yields the most preferable route and eliminates the need to use the selection procedure (21).

On the other hand, in the cases of large networks, the second algorithm requires more time for computations than the first algorithm based on the operator '$\wedge$'.

3.3. Most Reliable Route Interval Algorithm Based on '×' Operator for Acyclic Network

In this section, we shall consider an acyclic network with natural consecutive numbering of nodes from 1 to n, such that the number of any node j is greater than the number of all immediately preceding nodes linked by arcs to node j. The problem is to find the most reliable route from the source node (starting node 1) to a destination node t, $t \le n$.

Let P_j be the interval possibility from node 1 to node j, and $P_j = [\underline{p}_j, \overline{p}_j]$. By definition for the starting node 1,

P_1= [1.0,1.0]. The destination node is denoted by t, $t \le n$.

The interval values of P_j, $j = \overline{2,t}$ will be computed recursively using the formula

(23) $P_j = \max_{i \in N_j} \{P_i \times \pi_{ij}\}$

where i ranges over the set of all preceding nodes N_j. π_{ij} is the interval possibility between current node j and its predecessor i, and $\pi_{ij} = [\underline{\pi}_{ij}, \overline{\pi}_{ij}], \underline{\pi}_{ij} \ge 0, i \in N_j$.

To obtain the optimal solution of the problem, we will use the following label of node j:

(24) Node j Label = $\{[\underline{p}_j, \overline{p}_j], k1\}$

where $k1$ is the node immediately preceding j, which yields the maximum P_j.

The interval most reliable route algorithm consists of the following generalized steps:

***Step* 1**. Set j = 1. Assign to the source node (node 1) the label [[1.0, 1.0], –].

***Step* 2**. Set $j = j + 1$. Compute the possibility P_j to node j using the formula (23). Label the node j by using the labeling (24).

***Step* 3**. If $j = t$ go to *step* 4, else go to *step* 2.

***Step* 4**. Obtain the optimum route H between nodes 1 and t, starting from t and tracing backward through the nodes using label's information.

3.3.1. Analysis of the Complexity of the Most Reliable Route Interval Algorithm Based on '×' Operator for Acyclic Network

Consider the network in *figure 2*. The cardinality $|N_j|$ of the set of entering arcs N_j into node j is (j – 1), $|N_j| = j - 1$.

Each comparisons of two intervals $V = [\underline{v}, \overline{v}]$ and $W = [\underline{w}, \overline{w}]$ includes many comparisons and many additions.

$P_2 = \{P_1 \times \pi_{12}\} = \{[\underline{p}_1, \overline{p}_1] \times [\underline{\pi}_{12}, \overline{\pi}_{12}]\}$

$= [\underline{p}_1 \underline{\pi}_{12}, \overline{p}_1 \overline{\pi}_{12}] \Rightarrow$ We need only **2** multiplication to determine P_2.

$P_3 = max\{(P_1 \times \pi_{13}), P_2 \times \pi_{23})\}$, $\Rightarrow$ We need only **4** multiplications and only **1** comparison (of two intervals) to determine P_3.

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$P_{n-1} = max\{(P_1 \times \pi_{1(n-1)}), (P_2 \times \pi_{2(n-1)}), (P_3 \times \pi_{3(n-1)}), ...,$
$(P_{(n-2)} \times \pi_{(n-2)(n-1)})\} \Rightarrow$ We need only **(2*n* – 4)** multiplications and **(*n* – 3)** comparisons to determine P_{n-1}.

$P_n = max\{(P_1 \times \pi_{1n}), (P_2 \times \pi_{2n}), (P_3 \times \pi_{3n}), ...,$
$(P_{(n-1)} \times \pi_{(n-1)n})\} \Rightarrow$ We need only **(2*n* – 2)** multiplications and **(*n* – 2)** comparisons to determine P_n.

To obtain $P_j = [\underline{p}_j, \overline{p}_j]$ we need (j - 1) multiplications of nonnegative intervals and (j - 2) comparisons of intervals.

The total number of multiplications is $2 \times \sum_{j=2}^{n} (j-1)$, and the total number of comparisons is $\sum_{j=2}^{n} (j-2)$.

We set $\chi = j - 1$ and $\delta = j - 2$, and we obtain

The total number of multiplications is ξ,

$$\xi = 2 \times \sum_{\chi=1}^{n-1} \chi = (n\text{-}1)n$$

The total number of comparisons is ψ,

$$\psi = \sum_{\delta=1}^{n-2} \delta = \frac{(n-2) \times (n-1)}{2}$$

The running time of the algorithm is bounded by $O(multi = \xi, comp = \psi)$.

Numerical Example

Consider the network in *figure 5*. The parameters along the arcs are nonnegative interval possibilities π_{ij} of not being stopped between nodes i and j.

Using the algorithm described in section 3.3, the following

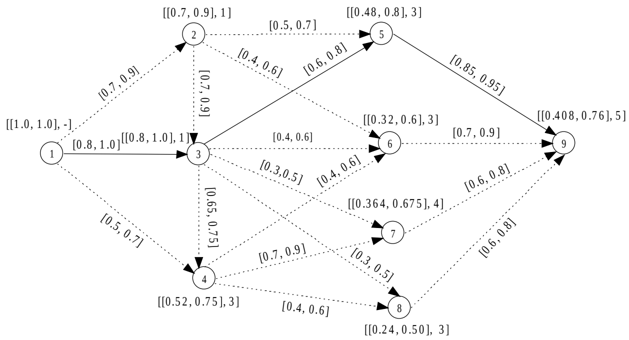

Figure 5

computational results are obtained:

P_1 = [1.0, 1.0], with label [[1.0, 1.0], -];

$P_2 = \{P_1 \times \pi_{12}\}$ = [0.7, 0.9], with label [[0.7, 0.9], 1];

$P_3 = \max_{i=1,2}\{P_i \times \pi_{i3}\}$ = max {[0.8, 1.0], [0.49, 0.81]} =

=[0.8, 1.0], with label [[0.8, 1.0], 1];

and so on.

The optimal solution is obtained by using label's information:

9 → [[0.408, 0.760], 5] → 5 → [[0.48, 0.80], 3] → 3 → [[0.80, 1.00], 1] → 1 → [[1.0,1.0], -].

Hence, the most reliable route is **1 → 3 → 5 → 9** with the corresponding interval possibility **[0.408, 0.760]**.

In *figure 5*, the solid lines indicate the most reliable route from the source node 1 to the destination node 9.

3.4. Most Reliable Route Algorithm Based on Interval Possibility for Acyclic Network

Consider an acyclic network with natural consecutive numbering of nodes from 1 to n, such that the number of any node j is greater than the number of all immediately preceding nodes linked by arcs to node j.

Assume that the possibilities π_{ij} are uncertain and known to fall within some intervals π_{ij}, $\pi_{ij} = [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$. The interval extension of (22) can be written as

(25) $P_j = \max_{i \in N_j} \{P_i \times \pi_{ij}\}$

where $P_j = [\underline{p}_j, \overline{p}_j]$, $P_i = [\underline{p}_i, \overline{p}_i]$, and '×' denotes the interval multiplications, and, by definition P_1 = [1.0, 1.0].

The formula (25) is represented in the form

$$\log P_j = \max_{i \in N_j} \{\log P_i + \log \pi_{ij}\}$$

where, if F denotes any of the intervals P_j, P_i or π_{ij}, $\log F = [\log \underline{F}, \log \overline{F}]$. Since $\log \pi_{ij} \le 0$, and $\log P_i \le 0$, maximizing this sum is equivalent to the following minimization problem

(26) $U_j = \min_{i \in N_j} \{U_i + D_{ij}\}$

where

(27) $U_j = -\log P_j$, $U_i = -\log P_i$, $D_{ij} = -\log \pi_{ij}$

The operation min {} is performed on the basis of (6) and/or (10), (11).

A more effective interval acyclic algorithm, using the midpoint and half-width notation (13), $R = [\, m(R), \Delta(R)]$, can be developed.

Let u_j denote the real shortest distance from 1 to node j. The real values u_j, $j = \overline{2,t}$ are computed using the recursive noninterval formula

(28) $u_j = \min_{i \in N_j} \{u_i + d_{ij}\}$

where d_{ij} is the midpoint of D_{ij}, $u_1 = 0$.

The following labeling of node j is used

(29) node j Label = $[u_j, k2, \Delta_{k2j}]$

where $k2$ is the node immediately preceding j and $k2 \in N_j$, that leads to the shortest distance u_j, and Δ_{k2j} is the half-width of D_{k2j}.

Further it is assumed that the network is described using interval notation with midpoint and half-width. It is also assumed a natural consecutive numbering of nodes from 1 to n, such that the number of any node i, $i \in N \mid 1$ is greater than the number of any immediately preceding node $k2$, $k2 \in N_i$, and where N is the set of nodes, $N = \{1, 2, ..., n\}$.

Hence, a preliminary step includes the conversion of interval possibilities to log interval possibilities, using (27), and then, the conversion of the usual interval notation in (26) to midpoint and half-width notation, using (13).

The algorithm consists of the following generalized steps:

***Step* 1**. Assign the label [0, –, 0] to the source node 1. Set $j = 1$.

***Step* 2**. Set $j = j + 1$. Compute the shortest distance from source node 1 to node j, by using recursive formula (28). Label node j by using (29).

If $j < t$ repeat *step* 2.

***Step* 3**. Obtain the optimum route H between nodes 1 and t, starting from node t and tracing backward through the nodes using the label's information.

***Step* 4.** Obtain the half-width $\Delta(U_t)$ of the interval solution U_t, adding the corresponding Δ_{ij} encountered along the optimum route H^*

$$\Delta(U_t) = \sum_{(i,j) \in H^*} \Delta_{ij}$$

***Step* 5.** Obtain the interval solution U_t, $U_t = [u_t - \Delta(U_t),\ u_t + \Delta(U_t)\ ]$.

***Step* 6**: Obtain the shortest route logarithmic interval length and convert back to non-logarithmic notation.

3.4.1. Analysis of the Complexity of the Most Reliable Route Algorithm Based on Interval Possibility for Acyclic Network

Consider an acyclic network in *figure 2*. The cardinality $|N_j|$ of the set of entering arcs N_j into node j is (j – 1), $|N_j|$ = (j – 1).

$u_2 = u_1 + d_{12} \Rightarrow$ We need only **1** addition to determine u_2.

$u_3 = min\ \{(u_1 + d_{13}), (u_2 + d_{23})\} \Rightarrow$ We need only **2** additions and **1** comparison to determine u_3.

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$u_{(n-1)} = min\ \{(u_1 + d_{1(n-1)}), (u_2 + d_{2(n-1)}), (u_3 + d_{3(n-1)}), \ldots, (u_{n-2} + d_{(n-2)(n-1)})\} \Rightarrow$ We need only **(n - 2)** additions and **(n - 3)** comparisons to determine $u_{(n-1)}$.

$u_n = min\ \{(u_1 + d_{1n}), (u_2 + d_{2n}), (u_3 + d_{3n}), \ldots, (u_{n-1} + d_{(n-1)n})\} \Rightarrow$ We need only **(n - 1)** additions and **(n - 2)** comparisons to determine u_n.

Hence, to obtain u_j we need (j – 1) additions and (j – 2) comparisons.

The number of additions is $\sum_{j=2}^{n} (j-1)$.

The number of comparisons is $\sum_{j=2}^{n} (j-2)$.

We set $\chi = j - 1$ and $\delta = j - 2$, and we obtain

$$\sum_{\chi=1}^{n-1} \chi = \frac{(n-1) \times n}{2}$$

$$\sum_{\delta=1}^{n-2} \delta = \frac{(n-2) \times (n-1)}{2}$$

The total number of additions is Γ, $\Gamma = \frac{(n-1) \times n}{2} + (n-1)$ (total half-width) + 2 additions to obtain the traditional interval representation + 2 additions to convert logarithmic interval length to non-logarithmic interval length.

The total number of comparisons is Φ,

$$\Phi = \sum_{\delta=1}^{n-2} \delta = \frac{(n-2) \times (n-1)}{2}.$$

Table 1 - Logarithmic transformation of interval possibilities

Road segment $(i,\ j)$	$\pi_{ij} = [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$	$\log \underline{\pi}_{ij}$	$\log \overline{\pi}_{ij}$	$[-1,-1]\times[\log\underline{\pi}_{ij},\ \log\overline{\pi}_{ij}]$	$[m(\log\pi),\ \Delta(\log\pi)]$
(1, 2)	[0.70, 0.90]	-0.1549	-0.0458	[0.0458, 0.1549]	[0.1004, 0.0546]
(1, 3)	[0.80, 1.00]	-0.0969	0.0000	[0.0000, 0.0969]	[0.0485, 0.0485]
(1, 4)	[0.50, 0.70]	-0.3010	-0.1549	[0.1549, 0.3010]	[0.2280, 0.0730]
(2, 3)	[0.70, 0.90]	-0.1549	-0.0458	[0.0458, 0.1549]	[0.1004, 0.0546]
(2, 5)	[0.50, 0.70]	-0.3010	-0.1549	[0.1549, 0.3010]	[0.2280, 0.0730]
(2, 6)	[0.40, 0.60]	-0.3979	-0.2218	[0.2218, 0.3979]	[0.3100, 0.0880]
(3, 4)	[0.65, 0.75]	-0.1870	-0.1249	[0.1249, 0.1870]	[0.1560, 0.0310]
(3, 5)	[0.60, 0.80]	-0.2218	-0.0969	[0.0969, 0.2218]	[0.1590, 0.0625]
(3, 6)	[0.40, 0.60]	-0.3979	-0.2218	[0.2218, 0.3979]	[0.3100, 0.0880]
(3, 7)	[0.30, 0.50]	-0.5228	-0.3010	[0.3010, 0.5228]	[0.4119, 0.1109]
(3, 8)	[0.30, 0.50]	-0.5228	-0.3010	[0.3010, 0.5228]	[0.4119, 0.1109]
(4, 6)	[0.40, 0.60]	-0.3979	-0.2218	[0.2218, 0.3979]	[0.3100, 0.0880]
(4, 7)	[0.70, 0.90]	-0.1549	-0.0458	[0.0458, 0.1549]	[0.1004, 0.0546]
(4, 8)	[0.40, 0.60]	-0.3979	-0.2218	[0.2218, 0.3979]	[0.3100, 0.0880]
(5, 9)	[0.85, 0.95]	-0.0705	-0.0222	[0.0222, 0.0705]	[0.0464, 0.0242]
(6, 9)	[0.70, 0.90]	-0.1549	-0.0458	[0.0458, 0.1549]	[0.1004, 0.0546]
(7, 9)	[0.60, 0.80]	-0.2218	-0.0969	[0.0969, 0.2218]	[0.1590, 0.0625]
(8, 9)	[0.60, 0.80]	-0.2218	-0.0969	[0.0969, 0.2218]	[0.1590, 0.0625]

So, the running time of the algorithm is limited by O(*addi* = Γ, *comp* = Φ).

If we develop the most reliable route algorithm (based on interval possibility for acyclic network) based on traditional interval representation, the complexity of the algorithm will be very high. To compare two intervals many comparisons and many additions are needed.

Numerical Example

Consider the network represented in *table 1*. Using (27) and (13) the network in *figure 6* is obtained.

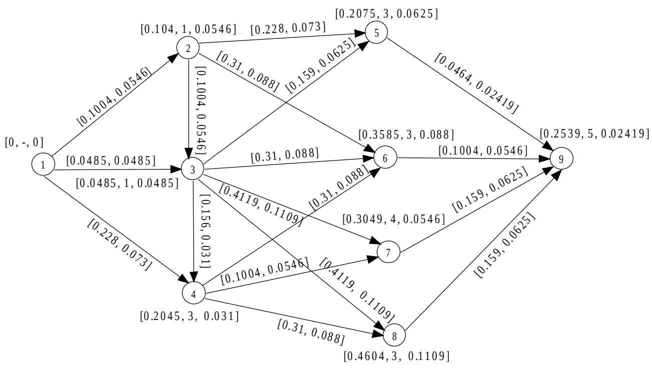

Figure 6

As an example, $\pi_{12} = [\underline{\pi}_{12}, \overline{\pi}_{12}] = [0.7,\ 0.9]$, then $\log\underline{\pi}_{12} = -0.1549$, $\log\overline{\pi}_{12}$ = -0.0458 and $[-1,-1]\times[\log\underline{\pi}_{12}, \log\overline{\pi}_{12}] = [0.0458$, 0.1549].

Using the midpoint and half-width notation (13) we obtain the equivalent representation of the interval, D_{12}, $D_{12} = -\log \pi_{12}$ = [0.1004, 0.0546] .

Using the algorithm we obtain the following results:

u_1 = 0, with label **[0, -, 0]**;

$u_2 = u_1 + d_{12}$ = 0 + 0.1004 = 0.1004, with label **[0.1004, 1, 0.0546]**;

$u_3 = \min_{i=1,2}\{u_i + d_{i3}\}$ =

= min $\{(u_1 + d_{13}),\ (u_2 + d_{23})\}$ =

= min $\{(0 + 0.0458), (0.1004 + 0.1004)\}$

= 0.0458, with label **[0.0485, 1, 0.0485]**;

and so on. All the computational results are summarized in *figure 6*.

Tracing backward using label's information the most reliable route is as follows:

9 $\Rightarrow$ [0.2539, 5, 0.02419] $\Rightarrow$ 5 $\Rightarrow$ [0.2075, 3, 0.0625] $\Rightarrow$ 3 $\Rightarrow$ [0.0485, 1, 0.0485]] $\Rightarrow$ 1 $\Rightarrow$ [0, -, 0].

The half-width of the optimal solution is:

$\Delta_9 = \Delta_{59} + \Delta_{35} + \Delta_{13}$ = 0.1352.

We have got the midpoint and half-width values, and now it is possible to obtain the interval possibility.

$U_9 = [\underline{u}_9,\ \overline{u}_9]$ = [0.1187, 0.3891].

$U_j = -\log P_j$;

Thus, $\log P_9$ = [-0.3891, -0.1187];

So, P_9 = **[0.408, 0.760]**.

The most reliable route is (1 $\Rightarrow$ 3 $\Rightarrow$ 5 $\Rightarrow$ 9), and the corresponding interval possibility are **[0.408, 0.760]**.

4. Conclusions

Four algorithms are proposed for solving the most reliable route problem in finite fuzzy networks. The uncertainty about the reliability of a route is represented in a possibilistic setting. The plausibility of not being stopped on a segment of the route is described using the corresponding possibility. The new algorithms maximize the possibility of not being stopped on the route between an origin node and a destination node. The analysis of the complexity of all five algorithms is evaluated.

The first and second algorithms are based on the usage of „and“ and „product“ operators to determine the strongest route, that is, the most reliable route in a finite fuzzy network. In the case of large networks, the second algorithm requires more time for computations than the first algorithm.

The third algorithm uses multiplication of interval possibilities and yields directly the largest interval possibility of not being stopped on the route. It is suitable for networks in which the time needed to obtain the solution itself is not a crucial factor.

The fourth algorithm for acyclic network is based on the concept of interval possibility. The transformation of the initial representation into logarithmic form is accomplished only once at the beginning and than the simple midpoint algorithm for solving interval acyclic algorithm.

The complexity of these new algorithms is evaluated, all algorithms are polynomial.

In [19, 20], the author proposed a method to solve the most reliable route problem with less degree of uncertainty. If the probabilities of a given network are with higher degree of uncertainty, i.e., the probabilities are given by upper and lower limits within which values are expected to fall, this method can not be used to solve the problem.

Numerical examples are given to illustrate the efficient assessment of the solution and the workability of the developed algorithms.

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Manuscript received on 23.09.2009

Akter Hossain *was born in Dhaka, Bangladesh. He received Master’s Engineering in Control Systems at Technical University of Sofia, Sofia (1999). He got Post-Graduation in Dynamic Modeling, Management and Engineering at Technical University of Sofia, Sofia (2002). At the present time he is waiting for PhD degree. His research interests are in the fields of decision-making, operations research, management.*

Contacts:
Ecole Doctorale
Filiure Francophone
Technical University of Sofia
Tel. 00359 887317573
e-mail: akter_h@yahoo.com; makter@gmail.com

Geo I. Gatev *received his Electrical Engineering degree from the Czech Technical University, Prague (1960), and his PhD degree in Technical Cybernetics from the Moscow Power Institute, Moscow (1968). He was Senior Research Fellow and vice-director in the Central Research Institute of Complex Automation, Sofia (1971-1986). Since 1988 was Professor at the Technical University of Sofia. His research and teaching interests was in the fields of control systems, decision-making, complex systems. He was the author of six books and over 200 papers. He was Senior Member of IEEE.*