Modeling and Analysis of Distributed Parameter System

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Key Words: *Partial differential equations; method of* **Green***'s functions; analytical models of distributed parameters systems; distributed characteristics in stationary and non-stationary regime.*

Abstract. *In this paper an analytical model of the temperature in* ***HVAC****-system for buildings conditioning based on partial differential equation of* ***Newton-Richman*** *is presented. A solution of the equation by using the* ***Green****'s functions method is submitted. On the basis of the solution analytically the stationary and non-stationary working states of modeled plant with distributed parameters – the temperature in* ***HVAC****-system are described. The dynamic system model is simulated in a change range of the basic parameters according to the accepted standards of comfort. Visualized and analyzed are the characteristics specificities, from which is derived the temperature in the building conditioning system as a typical control plant with distributed parameters.*

1. Introduction

The values, characterizing the climate in premises (residential, offices, industrial or warehouses) of ***HVAC*** (***Heating, Ventilating and Air-Conditioning***)***-system*** (*figure 1*) for building conditioning are:

- ***temperature*** *TA* of the supply air in the building;
- ***relative humidity*** *RH* of the air, related with *TA*;
- ***exchange rate*** (***velocity***) *VA* of the air in the building rooms.

Their values are conformable to the corresponding health and hygiene standards of comfort.

These parameters $TA(x,t)$, $RH(x,t)$, $VA(x,t)$ (*figure 1*) are also related with spatially distributed variables, which define the plant (air-conditioner) as a dynamic system with distributed parameters [1 ÷ 5]. The presented paper is intended to describe and model the temperature in the building conditioning system as controlled plant.

2. Plant Model

The controlled plant is presented with the corresponding:

- Structural model on *figure 2*, which covers not only the values listed above, but also some disturbances causing reparame–terization and restructuring;
- Generalized technological scheme of ***HVAC-system*** for conditioning by *TA*, *RH*, *VA* (*figure 3*), where the respective ***controlling*** (***input***) values are: for the temperature – the ***flow rate*** Q_{TA} of heat-transferring material (refrigerant) in the heater (chiller) and the fresh/exhaust air ratio γ, but as key and major is assumed the value Q_{TA}; for the relative humidity *RH* it is ***volumetric outgo*** Q_{RH} of the water aerosol in the condenser area and the ratio γ; for the exchange *VA* the ***turnovers*** Ω_{VA} of the fans, controlling the velocity of airflow circulating in the rooms.

As a basic output (controlled) value the temperature *TA* into the building rooms (as it is assumed that the values *RH* and *VA* are effectively stabilized) is considered, and as an input (controlling) value – the flow rate Q_{TA} of heat-transferring material (refrigerant) through the heater (chiller). This is included in the model on *figure 4*, which incorporates all other values in one general disturbance ς.

Known from the thermodynamics is the ***Newton-Richman*** law for forced convective heat transfer in non-stationary working state (1).

This equitation for heat transfer through heat conduction and forced convective, presented as (2), is assumed as an ***analytical model*** of process of temperature change in the ***HVAC****-system* in non-stationary state, where:

- $TA(x, t)$ is the fluid temperature;
- λ is heat conduction coefficient of the envi–ronment;
- v is fluid velocity;
- c is heat capacity;
- k is heat constant;
- ρ is fluid density;
- q is heat flow vector;
- Q is energy of the heat source.

The equivalent transformations of the equitation (2) result in a canonical class non-homogeneous differential equation with partial derivatives of second order (3), presented in general symbol description. The initial and limitation conditions for the solution of (3) are (4) and (5).

$$(1)\quad \rho\, c\,\frac{\partial TA(x,t)}{\partial t}+\nabla\left(-k\,\nabla TA(x,t)+\rho\, cv\, TA(x,t)\right)=Q$$

$$(2)\quad \lambda\,\frac{\partial^2 TA(x,t)}{\partial x^2}+q=\rho\, c\,\frac{\partial TA(x,t)}{\partial t}$$

$$(3)\quad u(x,t)\,\hat{\equiv}\, TA(x,t);\quad a\,\hat{\equiv}\,\sqrt{\frac{\lambda}{\rho c}};\quad f(x,t)\equiv\frac{q}{\rho c}-bu(x,0)$$

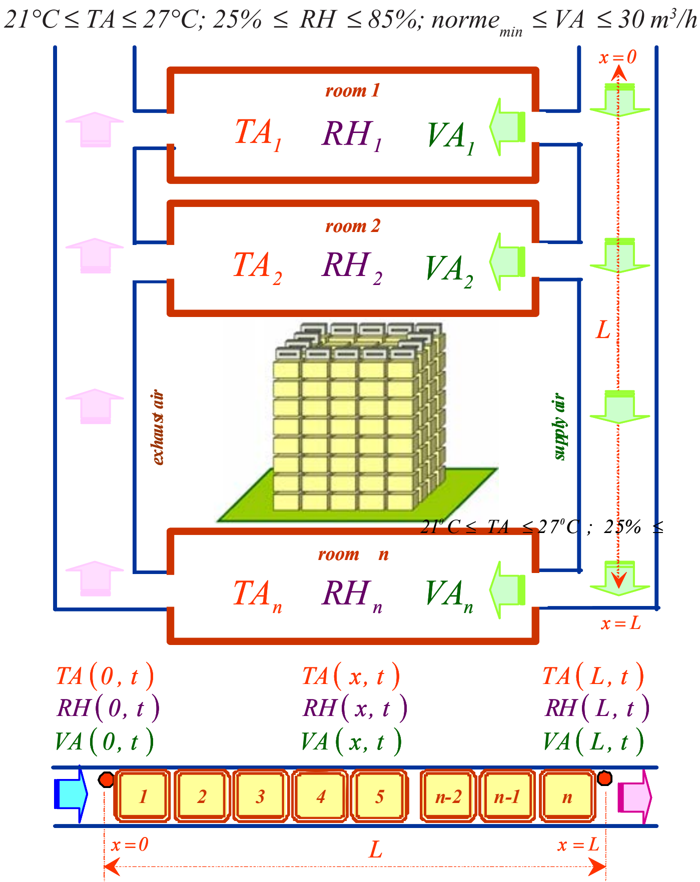


Figure 1

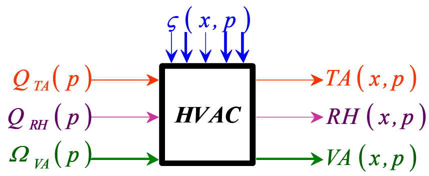


Figure 2

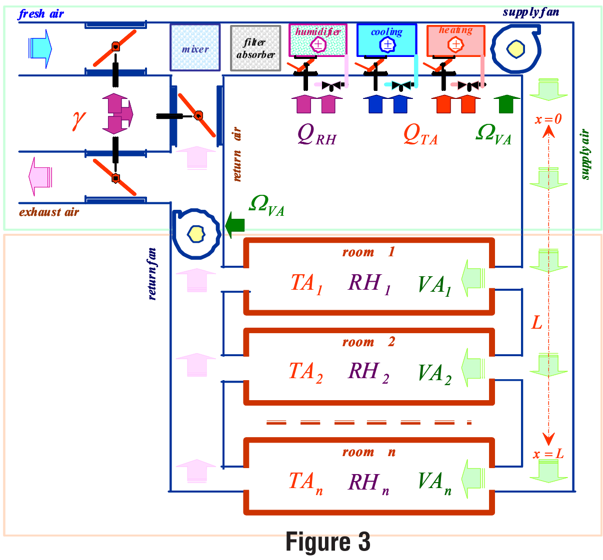


Figure 3

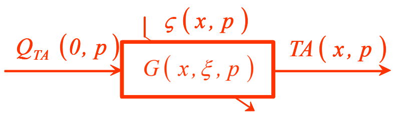


Figure 4

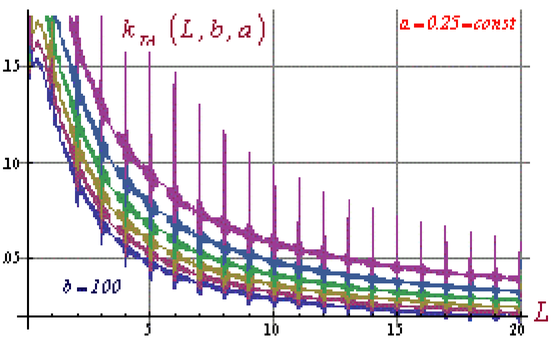


Figure 5

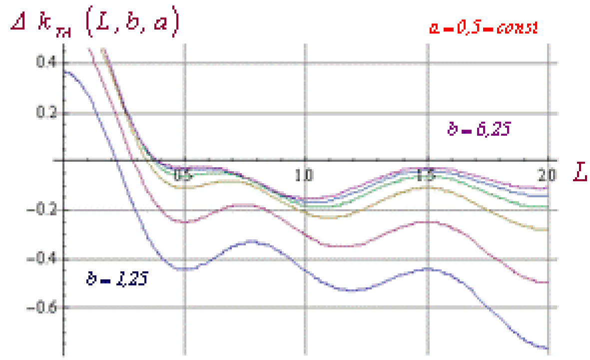


Figure 8

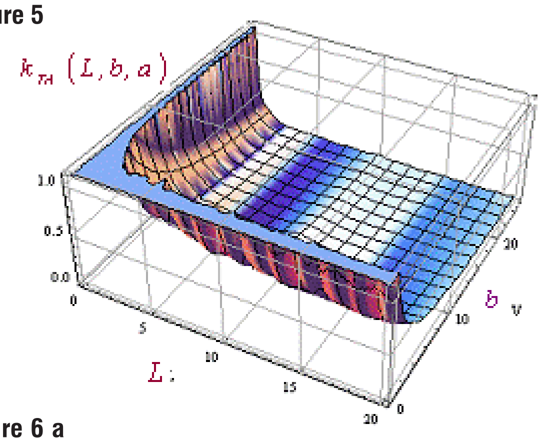


Figure 6 a

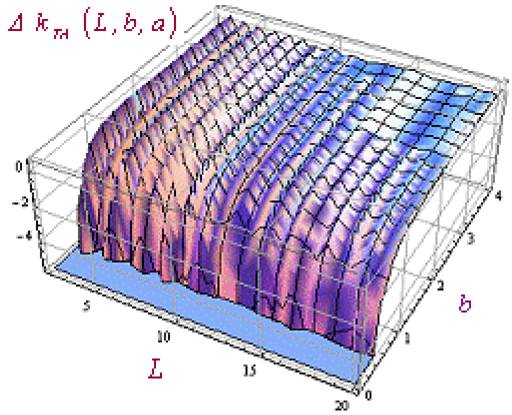


Figure 9 a

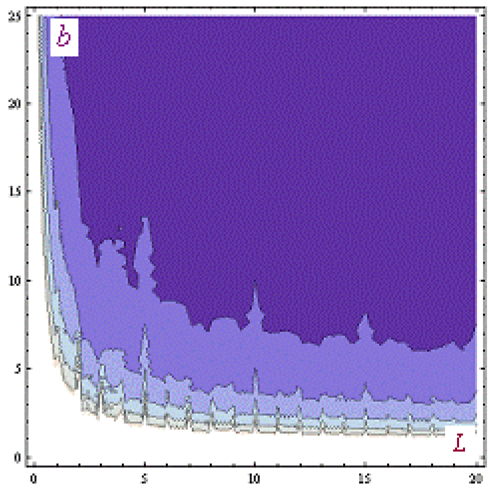


Figure 6 b

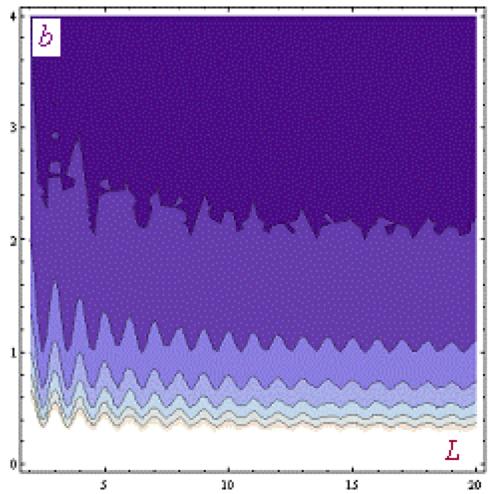


Figure 9 b

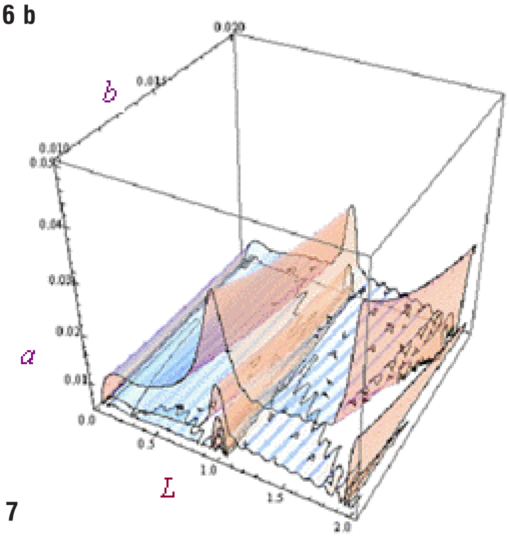


Figure 7

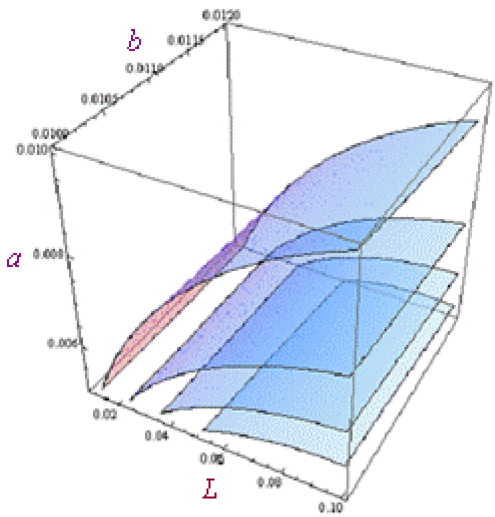


Figure 10

$$u(x,0) \hat{\equiv} TA(x,0) = u_0(x) \hat{\equiv} TA_0(x)$$

(4) $$\frac{\partial u(0,t)}{\partial t} \hat{\equiv} \frac{\partial TA(0,t)}{\partial x} = g_1(t), \quad \frac{\partial u(L,t)}{\partial t} \hat{\equiv} \frac{\partial TA(L,t)}{\partial x} = g_2(t)$$

(5) $$0 \le x \le l, \; t \ge 0, \; a \ne 0$$

(6) $$\varpi(x,t) = f(x,t) + u_0(x)\delta(t) - a^2\delta(x)g_1(t) + a^2\delta(L-x)g_2(t)$$

(7) $$G(x,\xi,t) = \frac{e^{-bt}}{L}\left(1 + 2\sum_{n=1}^{\infty}\left(\cos\frac{n\pi x}{L}\cos\frac{n\pi\xi}{L}e^{-\frac{n^2a^2\pi^2}{L^2}t}\right)\right)$$

(8) $$G_{TA}(x,\xi,p) = \frac{TA(x,p)}{Q_{TA}(x,p)} = \frac{1}{(Lp+b)} + \frac{2}{L}\sum_{n=1}^{\infty}\frac{\cos n\pi x L^{-1}\cdot\cos n\pi\xi L^{-1}}{(n^2a^2\pi^2L^{-2}+p+b)}$$

(9) $$TA(x,t) \hat{\equiv} u(x,t) = \int_{t_0}^{t}\int_{D} G(x,\xi,t,\tau)\cdot\varpi(\xi,\tau)\,d\xi\,d\tau$$

(10) $$G_{TA}(x,\xi,j\omega) = \frac{TA(x,j\omega)}{Q_{TA}(x,j\omega)} = \frac{1}{(Lj\omega+b)} + \frac{2}{L}\sum_{n=1}^{\infty}\frac{\cos n\pi x L^{-1}\cdot\cos n\pi\xi L^{-1}}{(n^2a^2\pi^2L^{-2}+j\omega+b)}$$

(11) $$k_{TA}(L,b,a) = \frac{TA(L,b,a)}{Q_{TA}(L,b,a)} = b^{-1} + \frac{2}{L}\sum_{n=1}^{\infty}\frac{\cos n\pi x L^{-1}\cdot\cos n\pi\xi L^{-1}}{(n^2a^2\pi^2L^{-2}+b)},$$
$$a = a(\lambda,\rho,c);\; b = b(q,\rho,c)$$

(12) $$0{,}01 \le b \le 50;\; 0{,}005 \le a \le 0.65;\; 0 \le L \le 20$$

(13) $$\Delta k_{TA}(L,b,a) = \frac{\partial k_{TA}(L,b,a)}{\partial L}\Delta L + \frac{\partial k_{TA}(L,b,a)}{\partial b}\Delta b + \frac{\partial k_{TA}(L,b,a)}{\partial a}\Delta a$$

(14) $$\frac{\partial k_{TA}}{\partial L} = \frac{4a^2\pi^2\cos(L\pi)^2}{L^4\left(\frac{a^2\pi^2}{L^2}+b\right)^2} - \frac{2\cos(L\pi)^2}{L^2\left(\frac{a^2\pi^2}{L^2}+b\right)^2} - \frac{4\pi^2\cos(L\pi)\sin(L\pi)}{L\left(\frac{a^2\pi^2}{L^2}+b\right)^2}$$

(15) $$\frac{\partial k_{TA}}{\partial b} = \frac{1}{b^2} - \frac{2\cos(L\pi)^2}{L^2\left(\frac{a^2\pi^2}{L^2}+b\right)^2}$$

(16) $$\frac{\partial k_{TA}}{\partial a} = -\frac{4a\pi^2\cos(L\pi)^2}{L^3\left(\frac{a^2\pi^2}{L^2}+b\right)^2}$$

(17) $$\Delta k_{TA}(L,b,a) = \frac{4a^2\pi^2\cos(L\pi)^2}{L^4\left(\frac{a^2\pi^2}{L^2}+b\right)^2} - \frac{2\cos(L\pi)^2}{L^2\left(\frac{a^2\pi^2}{L^2}+b\right)^2} - \frac{4\pi^2\cos(L\pi)\sin(L\pi)}{L\left(\frac{a^2\pi^2}{L^2}+b\right)^2} + \\ + \frac{1}{b^2} - \frac{2\cos(L\pi)^2}{L^2\left(\frac{a^2\pi^2}{L^2}+b\right)^2} - \frac{4a\pi^2\cos(L\pi)^2}{L^3\left(\frac{a^2\pi^2}{L^2}+b\right)^2}$$

The standardizing function ϖ ***, the Green-function*** G ***, the transfer function*** G_{TA} ***and the solution*** u of the problem (with the specified initial and boundary conditions (4), (5)) are presented by the relations (6) $\div$ (9).

The frequency characteristics corresponding to the solution is (10). The transfer function $G_{TA}(x,\xi,p)$ of the modeled system (2) indicates that the system's dynamics is approximated by using sequentially-parallel connection of ambiguous, non-linear, inertial units from distributed type.

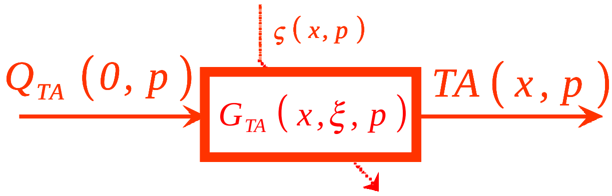


Figure 11a

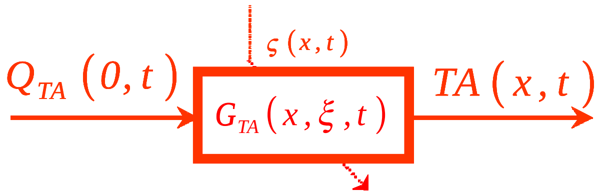


Figure 11b

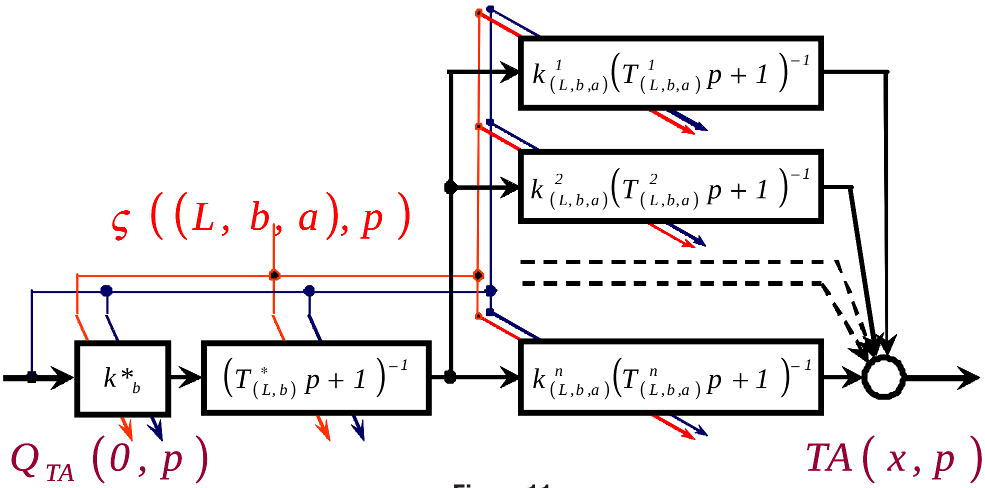


Figure 11c

3. Analysis of the Stationary State of Temperature in Conditioning System

The analytical model (1) determines the process of change in the temperature TA of circulating air in ***HVAC***-system as a static, ambiguous, non-linear, inertial dynamic system with distributed parameters. The steady state of the temperature TA ($t \to \infty$; $p \to 0$) is defined by the gain of the dynamic system (1). To determine the coefficient analytically, the distributed transfer function (8) is transformed equivalently to (11).

The gain k (L, ρ, q, k, λ, c) of the distributed dynamic system is a function of: the fluid density ρ; the heat constant k; the heat flow vector q; the heat conduction coefficient of the environment λ; the heat capacity c.

For the particular operating values (12) the stationary state of temperature (11) is modeled and simulated for $n = 20$. The results are presented in *figure 5* and *figure 6*. There is an obvious trend in the stationary state of decreasing the value of the gain k_{TA} (L, b, a) while: increase conditioning volume L and/or decrease a and/or decrease b. On *figure 6 b* a parametric plot of k_{TA} (L, b, a) is shown, which presents by colors these combinations of the parameters L, b, a, for which the coefficient value is constant.

By using ***3D***-parametric plot on (*figure 7*) the rate (trend) of change of the distributed system gain while changing the parameters L, b, a values is performed.

The sensitivity Δk_{TA} (L, b, a) of gain k_{TA} (L, b, a) to the parameters L, b, a in stationary state of the temperature TA is defined according to (13) ÷ (17). For particular change range of the values of parametric set L, b, a(12), Δk_{TA} (L, b, a) is modeled and simulated for $n = 20$ in (11), and results are presented in *figure 8* and *figure 9*. In *figure 10* the rate of change of gain sensitivity to the parameters L, b, a by ***3D***-parametric plot is presented.

4. Analisys of Non-stationary State of Temperature in Conditioning System

The relations (1) ÷ (10) describe the specifics of the non-stationary state of the temperature in the ***HVAC*** - system. The structural models of the distributed system presented in *figure 11* are assumed. For the particular change range of operating conditions (12) and simulated the dynamics of the temperature TA is modeled as a plant with distributed parameters for $n = 20$ in (8) ÷ (10). The dependencies modeling the temperature are simulated. Simulation results of the differential equitation (1) solution are presented by:

- ***The distributed step response function*** h_{TA}(18) (*figure 12*) describes the temperature changes for a single impulse input signal Q_{TA}(19) provided (20). The signification of characteristics h_{TA} is theoretical. In real terms it is not possible the requirements of input signal (19) to be covered. The distrib-

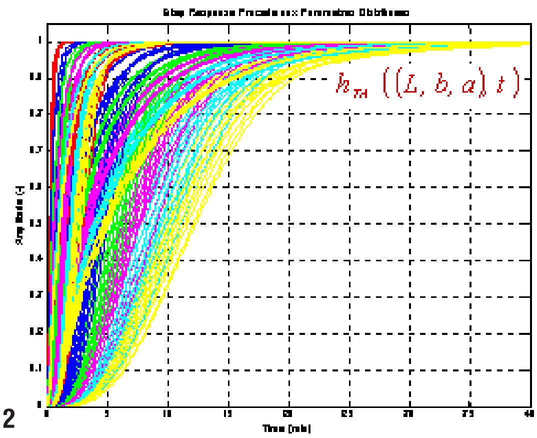


Figure 12

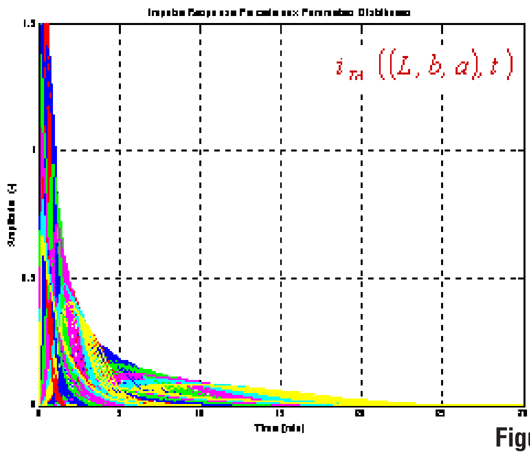


Figure 13

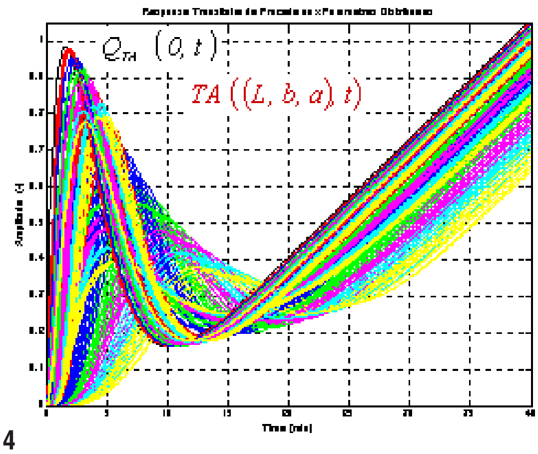


Figure 14

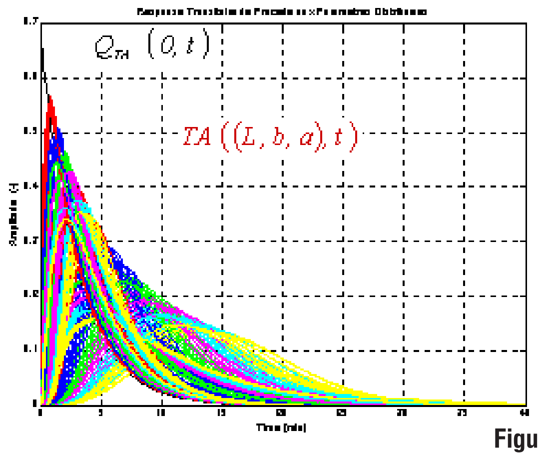


Figure 15

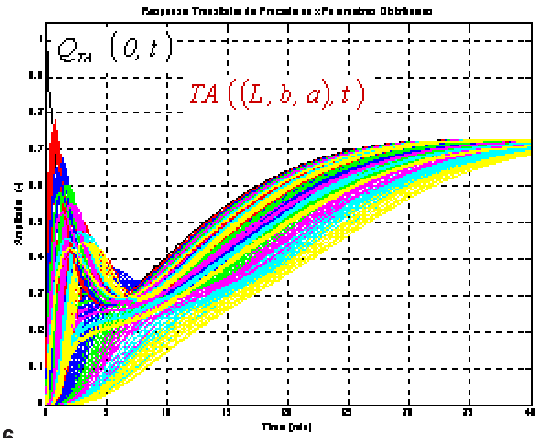


Figure 16

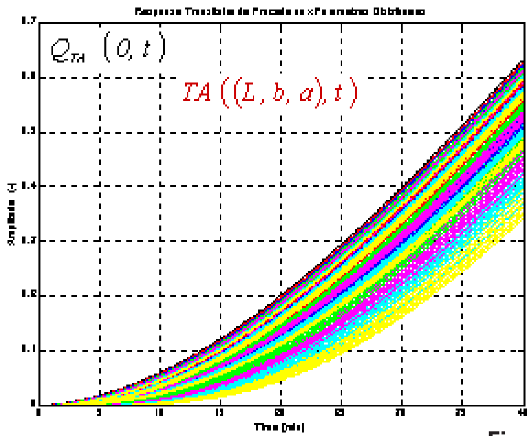


Figure 17

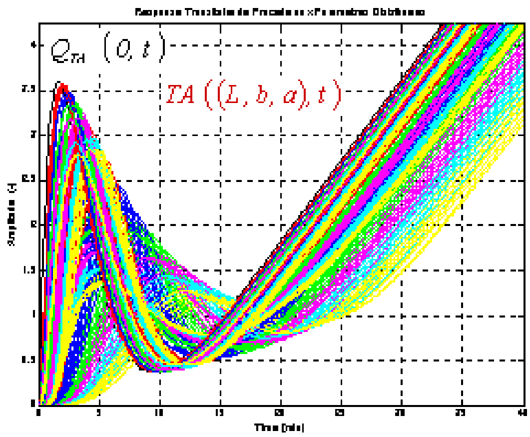


Figure 18

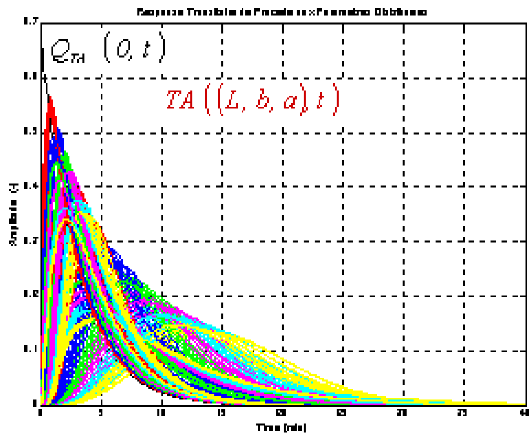
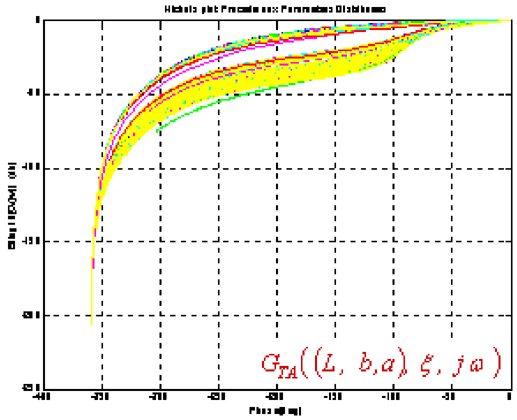


Figure 19



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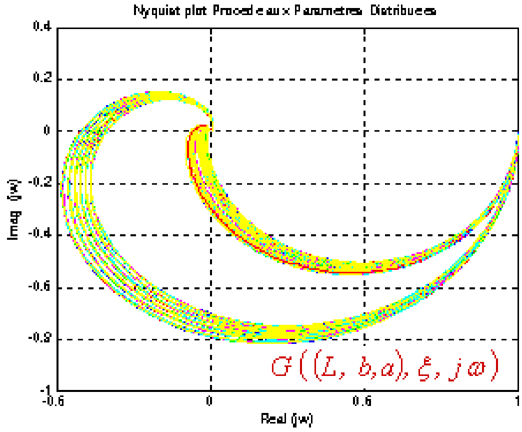


Figure 21

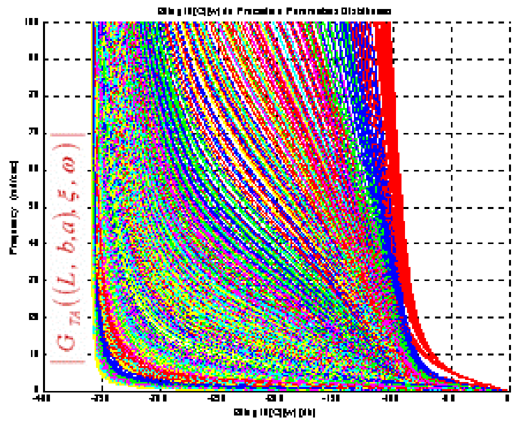


Figure 22

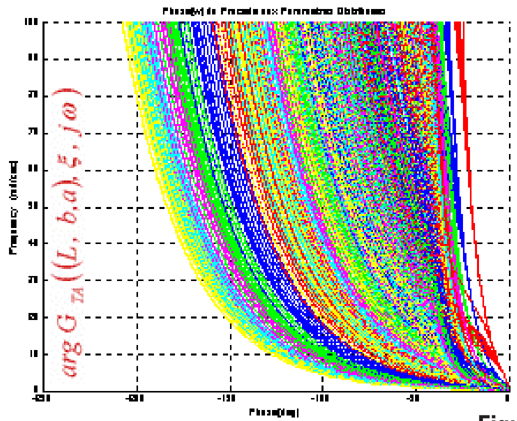
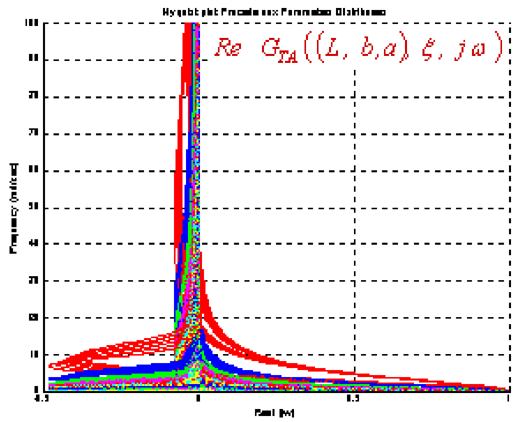


Figure 23



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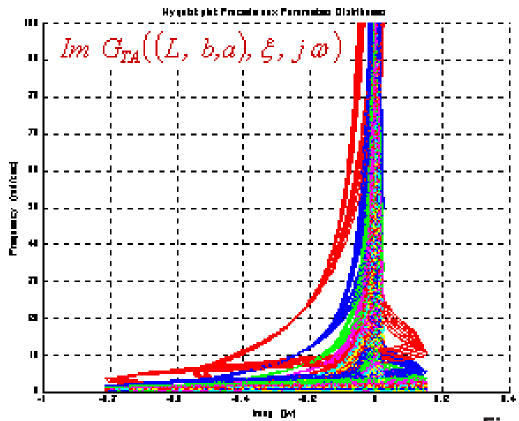

Figure 25

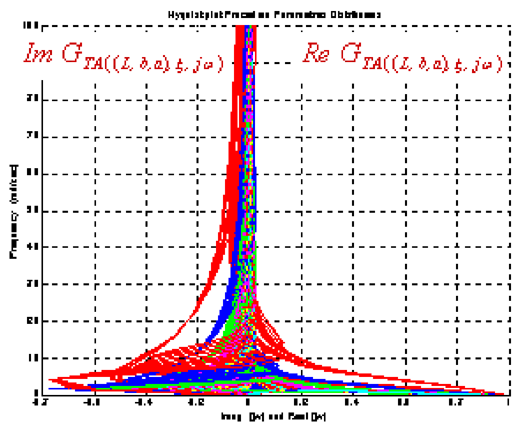

Figure 26

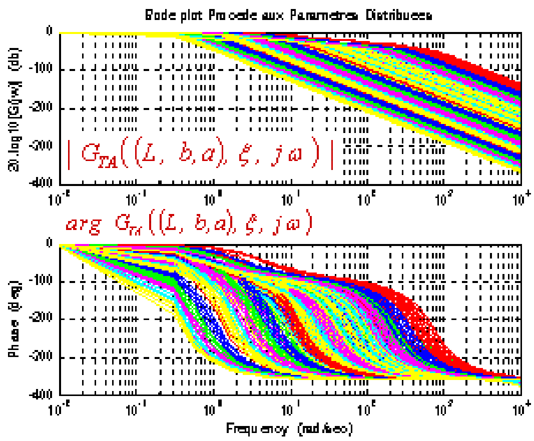

Figure 27

(18) $$h_{TA}(t, L, b, a) \hat{\equiv} TA(t, L, b, a)$$

(19) $$Q_{TA}(t) = 1(t)$$

(20) $$0{,}01 \le b \le 50;\ 0{,}005 \le a \le 0.65;\ 0 \le L \le 20$$

(21) $$h_{i,j,\kappa}(t), (i(L) \in [0, 20], j(b) \in [0.01, 50], \kappa(a) \in [0{,}005, 0.65])$$

(22) $$Q_{TA}(t) = \sin \omega t,\ Q_{TA}(j\omega) = \omega / ((j\omega)^2 + \omega^2)$$

(23) $$G(L, b, a, j\omega) \hat{=} \frac{TA(L, b, a, j\omega)}{Q_{TA}(j\omega)}$$

(24) $$G_{TA}(L, b, a, j\omega) \hat{\equiv} \left| G_{TA}(L, b, a, \omega) \right| e^{-j \arg(G_{TA}(L, b, b, \omega))}$$

(25) $$G_{\kappa, l, m}(t), (\kappa(L) \in [0, 20], l(b) \in [0.01, 50], m(a) \in [0.005, 0.65])$$

(26) $$G_{TA}(L, b, a, j\omega) = Re\, G_{TA(L,b,a)}(\omega) + j\ Im\ G_{TA(L,b,a)}(\omega)$$

(27) $$\left| G_{TA}(L, b, a, \omega) \right| = \sqrt{Re^2 G_{TA(L,b,a)}(\omega) + Im^2 G_{TA(L,b,a)}(\omega)}$$
$$\arg(G_{TA}(L, b, a, \omega)) = -\operatorname{arc\,tg} \frac{Im\, G_{TA(L,b,a)}(\omega)}{Re\, G_{TA(L,b,a)}(\omega)}$$

uted function, denoted as h_{TA} (*figure 12*), represents the family of all possible in the range (20) simulated in parallel impulse response functions $h_{i,j,k}$ (21) with „concentrated“ and „frozen“ in value parameters. In this sense h_{TA}, determined on the basis of (9), represents the temperature changes as a function of time t for every possible value of L, b, and a in accordance with the corresponding initial, limitation and technological terms (4), (5), (20). The solution of equitation (1) provides the opportunity the referred ***HVAC***-system to be modeled, simulated and analyzed for random set of terms (20) and particular specifics.

- ***The distributed impulse response function*** i_{TA} ((L, b, a), t) is shown in *figure 13*.
- ***The distributed transient characteristics*** (*figure 14÷ figure 19*) TA of the temperature, representing the system response to a random input signal Q_{TA}, plotted with bold dark line. These are real characteristics, because of the operational rational character of input signals and they represent changes of temperature TA as a function of time t for every possible value of L, b and a in accordance with the corresponding initial, limitation and technological terms (4), (5) (20) of the defined input signals Q_{TA}. Used signals Q_{TA} with typical forms are, which represent specific operational modes. The distributed frequency response (10) for particular real conditions (12), (20), as a model with distributed parameters of the temperature dynamics in ***HVAC***-system, is simulated. The simulation results of (10) with harmonic sinusoidal input signal Q_{TA} (22) are presented by.
- ***The distributed Nichols- plot*** G_{TA} (23) in *figure 20*, and its components - magnitude-frequency and phase-frequency plots (24) of system response as function of frequency ω are presented separately on *figure 22* and *figure 23*. By them the distributed frequency response represents the physical response of the analyzed distributed system to the changes in the gain and to the inertia of the temperature as a real dynamic system (24) response to the input signal Q_{TA} (22) for different values of the frequency ω. The distributed frequency response, here denoted as G_{TA}, represents the family of all possible in the range (20) simulated in parallel responses $G_{k,l,m}$ (25) with „concentrated“ and „frozen“ in value parameters. In this sense the distributed frequency response G_{TA}, determined on the basis of the dependency (10), represents the temperature TA changes versus Q_{TA} as a function of the frequency ω for every value of L, b and a *an* accordance with the corresponding initial, limitation and technological terms (4), (5), (20).
- ***The distributed Nyquist-plot*** (*figure 21*) $G_{TA}(L, b, a, j\omega)$ of the system's frequency response (10). Its real Re and imaginary Im components of (26) are presented in *figure 24 figure 26*. Their relation with the magnitude and phase (24) (magnitude-frequency and phase-frequency plots) is represented by (27).
- ***The distributed Bode-plot*** $G_{TA}(L, b, a, j\omega)$ of the system's frequency response (10) (*figure 27*), by its components - magnitude and phase (24). Both shown in parallel in *figure 27* – represent the physical reaction of the analyzed distributed system to the gain changes and the inertia of the temperature as real dynamic system to the input signal for different values of the frequency ω.

5. Conclusion

The basic values $TA(x,t)$, $RH(x,t)$, $VA(x,t)$ characterizing the climate in the premises are correlated and spatially distributed variables, which determine the climate as a dynamic system with distributed parameters. The main achievements of present article impose the following basic conclusions and results:

- The analytical model of the plant (the process of temperature TA change in ***HVAC***-system) (1)÷(10) represents, that the system's dynamics can be successfully approximated by using a sequentially-parallel connection of ambiguous, non-linear, inertial units from distributed type.

- From the stationary state simulations results (*figure 6* and *figure 7*) it is obvious that the trend of distributed system gain $k_{TA}(L, b, a)$ is decreasing, while the conditioned volume L increases (or while moving away the discussed position x in the area from the ***HVAC***-system energy source).
- The presented, modeled and simulated ***Nichols-, Nyquist-*** and ***Bode-plots of the system's frequency response*** illustrate systematically all specific characteristics of the distributed dynamic system (temperature) response of input signals as Q_{TA} (22).
- ***The distributed frequency responses*** (*figure 20 ÷ figure 27*), together with the ***distributed step response*** and ***transient characteristics*** (*figure 12 ÷ figure 19*), representing the modeled system response versus time t, determine the ***non-stationary state of temperature changes in the HVAC-system*** (1) ÷ (10) and (18) ÷ (27) as a static, ambiguous, inertial dynamic system with distributed parameters.

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