

Inverted Pendulum Control: an Overview

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Key Words: Cart – pendulum system; inverted pendulum; swing up control; local stabilization.

Abstract. This paper considers the problem of inverted pendulum control. Position control of the inverted pendulum in upright equilibrium state can be divided into two tasks: swing up control of the pendulum in upright position and local stabilization around the equilibrium point. Two main approaches for solving the first problem are presented: the energy approach and the speed gradient approach. At the same time, many modern methods for control of the inverted pendulum are also introduced. The presented methods solve the inverted pendulum swing up problem and ensure its global stabilization.

1. Introduction

Laboratory exercises play an important role in control theory and control engineering courses. The relation between the theoretical knowledge obtained in theoretical control courses and its practical application and implementation in laboratory experiments is a major part of contemporary education in the field. Recently, quite popular have become laboratories based on physical models of real devices and processes which are controlled by microprocessor regulators and programmable logic controllers (PLC) [24,26]. These are the so – called open laboratories, where the same equipment is used for carrying different experiments. A single well equipped laboratory supports most of the courses in dynamical systems analysis and control systems design. Laboratory models in such laboratories use PCs with standard input/output interface cards for data acquisition and control.

A major part of these laboratory models are built to represent certain characteristics and properties of the existing industrial processes and systems. However, there is a large group of models not having direct link to real systems, but nevertheless serve as a test bed for a variety of control algorithms. The typical example of such a device is the physical model of the inverted pendulum. The cart-pendulum system is one of the most popular laboratory models for practical implementation and demonstration of control systems. The inverted pendulum is a classical electromechanical device for testing some complicated system analysis and design methods. The purpose for exploring the cart – pendulum system is to represent the difficulties to control an inherently unstable plant, containing numerous nonlinearities, characterized by many equilibrium points and serving as an example for the fundamental structural limitations of using the feedback connection.

The cart-pendulum system in *figure 1* consists of the following parts [25]: *i)* mechanical part consisting of a cart driven by a DC motor and a two-pole pendulum attached to the cart, *ii)* I/O board with built in ADC and DAC, *iii)* power board with built – in amplifier, *iv)* personal computer with the appropriate software tools for connecting to the hardware. The cart can move

along a horizontal rail and the pendulum is able to rotate freely in a vertical plane parallel to the rail. In order to swing up or balance the pendulum around its equilibrium points, the cart has to move back and forth on the rail by a plane DC motor. The position of the cart on the rail and the angle of the pendulum are measured by two optical encoders.



Figure 1. The inverted pendulum laboratory model

The goal of the control algorithm is by using several oscillations with increasing amplitude to bring the pendulum poles around the upper equilibrium position without letting the angle and velocity become too large. After reaching this state the pendulum is stabilized there while allowing to move the cart along the rail.

2. Inverted Pendulum System Modeling

There are several cases for describing the inverted pendulum control problem in the control literature: *i)* inverted pendulum with fixed end point (pendulum of Furuta) [11,12,13,20,29,38,40], *ii)* inverted pendulum with moving cart [8,14,23,25,27,28,33, 34,42,43,44,46], *iii)* double inverted pendulum with fixed end point (acrobot) [1,17,39], *iv)* double inverted pendulum with moving cart [15,18,22,47], *v)* triple inverted pendulum with fixed end point [4,19,35]. The most popular case is the inverted pendulum with or without moving cart.

The model of the cart – pendulum system can be derived basically in two different ways: *i)* by using the formulation of Newton – Euler that leads to effective computing of the control law in real time [33,21,30,17,28] and *ii)* by using the formulation of Lagrange that is based on algebraic calculations over energy quantities using generalized coordinates and generalized forces [33,16]. Let us consider the free body diagram of the cart-pendulum system shown in *figure 2*. The cart moves along a horizontal rail and the pendulum rotates in a vertical plane

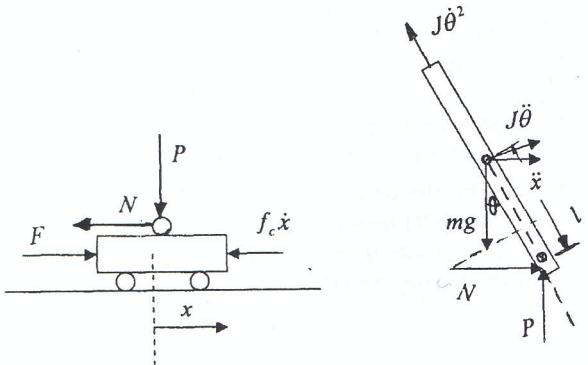


Figure 2. Free body diagram of the cart-pendulum system

parallel to the rail. The cart-pendulum system can be described as a fourth order dynamical system with state variables defined as follows: translational deviation x and translational velocity of the cart \dot{x} , angular deviation θ and angular velocity of the pendulum $\dot{\theta}$ measured in clock-wise direction. The input signal is the horizontal force applied to the cart. We introduce the following notation: M – the cart mass, m – the pendulum mass, l – the length from the axis of rotation to the pendulum center of mass, J – the pendulum moment of inertia with respect to the axis of rotation, f_c – the coefficient of viscous friction of the cart, f_p – the coefficient of viscous friction of the pendulum, F – the force acting on the cart, g – the gravitational constant, N – the reaction force of the pendulum on the cart in horizontal direction, P – the reaction force of the pendulum on the cart in vertical direction. The dynamical equations of the cart-pendulum system are given as follows [25,33,16]:

$$(1) \quad (M+m)\ddot{x} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 + f_c\dot{x} = F$$

$$(2) \quad (J+ml^2)\ddot{\theta} - mgl\sin\theta + ml\cos\theta\ddot{x} + f_p\dot{\theta} = 0.$$

Equation (1) describes the system motion in horizontal direction and equation (2) describes the rotational motion of the pendulum. Similarly are derived the dynamical equations of the double inverted pendulum by using the formulation of Newton-Euler [15,39,47] or by the formulation of Lagrange [18,22], as well as the equations of the triple inverted pendulum [19,35].

3. Main Approaches for the Inverted Pendulum Swing up Control

The inverted pendulum control problem consists of two main parts: *i)* swinging up the pendulum to the upright position and *ii)* stabilizing it in this position once it is reached there [3]. The solution of the second part is due to the fact that the desired upright position is a saddle point (unstable equilibrium) and once it is reached, due to small disturbances, the pendulum will go away from this position. The main approach for solving the first part of the problem is based on energy considerations [11,17]. Let us consider only the rotational motion of the pendulum, neglect the friction and assume that the linear acceleration determines the control signal u to the system. Measuring the moment of inertia with respect to the axis of rotation, the follow-

ing simplified dynamical equation of rotational motion for the pendulum can be obtained [11]:

$$(3) \quad J\ddot{\theta} = mlg\sin\theta - mlu\cos\theta.$$

The system has two state variables, the angle θ and the angular velocity $\dot{\theta}$. It is natural to choose the state space to be a cylinder. The behavior of the inverted pendulum under energy control is fully characterized by the normalized maximum acceleration u_{\max} and the acceleration of gravity g [11]. Many problems can be solved by controlling the energy of the pendulum instead of controlling its position and velocity directly. The energy of the pendulum from (3) when the control signal is zero ($u = 0$) is determined by the expression

$$(4) \quad E = \frac{1}{2}J\dot{\theta}^2 + mgl(\cos\theta - 1)$$

and the energy derivative when $u \neq 0$ is determined by

$$(5) \quad \frac{dE}{dt} = J\dot{\theta}\ddot{\theta} - mgl\dot{\theta}\sin\theta = -mul\dot{\theta}\cos\theta.$$

In order to increase energy, the pendulum acceleration sign should be opposite to the sign of the quantity $\dot{\theta}\cos\theta$. A control strategy is easily obtained by applying the method of Lyapunov with a Lyapunov function candidate

$$(6) \quad V = \frac{(E - E_0)^2}{2},$$

where E_0 is the energy level at equilibrium. The control law is determined as follows

$$(7) \quad u = k(E - E_0)\dot{\theta}\cos\theta.$$

The goal is that pendulum energy converges to the desired energy value E_0 . In order to change energy level faster the control signal should be as large as possible

$$(8) \quad u = ngsign\{(E - E_0)\dot{\theta}\cos\theta\},$$

where $ng = u_{\max}$ is the maximum value of the control signal. In order to avoid chattering the control law is determined as follows

$$(9) \quad u = sat_{ng}\{k(E - E_0)sign(\dot{\theta}\cos\theta)\},$$

where sat_{ng} is a linear function saturated at the value of ng . The control law (9) is similar to (7) when the error is small and is similar to (8) when the error is large. The parameter n determines the maximum value of the control signal and therefore the maximum rate of change of the pendulum energy. The parameter k determines the zone of linear motion of the pendulum. The pendulum energy in upright position is zero and therefore the desired energy level is $E_0 = 0$. The strategy for

swinging up the pendulum in upright position is characterized by the following two quantities: *i*) the number of swings which the pendulum makes before reaching the upright position and *ii*) the number of switches of the control signal.

Another approach for swinging up the pendulum is based on the speed gradient method [20,38]. The control strategy is based on the assumption that the upright equilibrium point has minimal energy and is the unique attracting point of the closed – loop system. Under certain conditions the speed gradient – energy algorithm stabilizes any specified energy level with arbitrarily small control signal. It is shown in [38] that the problem of global stabilization of the unstable equilibrium can not be solved by continuous static state feedback. The condition for existence of almost global point of attraction, i.e. attraction from all initial conditions except some set of zero measure, is that the energy level corresponding to the upright equilibrium point contains only one ω – limit point. For reaching the upright equilibrium point it is necessary to have either time – varying term or switching term in the regulator. Simple modification based on the variable structure system approach makes possible to achieve global stabilization of the unstable equilibrium position. Assume that the pendulum energy is given by (4), the Lyapunov function – by (6) and the equation of motion is described by $\dot{x} = f(x) + g(x)u$. The control law swinging up the pendulum in upright equilibrium position is computed as

$$(10) \quad u = -kg^T(x) \frac{\partial V(x)}{\partial x}.$$

After substitution in the Lyapunov function expression V the control signal is obtained as:

$$(11) \quad u = kE(\dot{\theta} \cos \theta),$$

where k is chosen as large as possible.

4. Methods for Control of the Inverted Pendulum Based on Global Stabilization

The inverted pendulum is an underactuated mechanical system that has fewer control inputs than degrees of freedom. The underactuated systems often contain structural properties that makes it difficult to apply standard nonlinear control methods such as complete feedback linearization and others. A swing up controller for a double inverted pendulum that is based on partial feedback linearization and passivity control together with energy shaping is developed in [47]. The system is transformed into the form

$$(12) \quad \dot{x} = Ax + Bu,$$

$$\xi = f(\xi) + g(x, \xi)u,$$

where the control law u is chosen as

$$(13) \quad u = -k_1x - k_2\dot{x} + k_3\tilde{u}.$$

The signal \tilde{u} is calculated by using the passivity property as follows:

$$(14) \quad \tilde{u} = E(k\dot{\theta} \cos \theta).$$

The first two components of (13) regulate the motion of the cart and the third component regulates the total energy and stabilizes the pendulum in the upright equilibrium position. The passivity property of the pendulum is used in [28] to stabilize its motion around the homoclinic orbit, i.e. this orbit that has zero energy level. The swing up control problem is solved by bringing the system trajectory towards its homoclinic orbit. The passivity property of the system suggests using the total energy in the controller design. The following Lyapunov function candidate is chosen

$$(15) \quad V(x, \theta) = \frac{1}{2} [k_1E(\theta, \dot{\theta})^2 + k_2x^2 + k_3\dot{x}^2].$$

Based on this Lyapunov function and the stability condition for the inverted pendulum, the following control law is proposed

$$(16) \quad u = \frac{k_3 \sin \theta (g \cos \theta - \dot{\theta}^2) - (1 + \sin^2 \theta)(k_2x + k_3\dot{x})}{k_3 + k_1E(1 + \sin^2 \theta)},$$

where k_d is a constant which is calculated from the stability condition for the inverted pendulum. In [28] it is shown that by using the control law (16) and with strictly positive constants k_1, k_2, k_3 and k_d , the solution of the closed – loop system converges to the homoclinic orbit containing the upright equilibrium point. By applying the energy approach, the inverted pendulum will reach the desired equilibrium point without being locally stable. Therefore, small disturbances will cause the pendulum to move out from the upright equilibrium state. The reason for this is that the upright pendulum position is a saddle point of mechanical energy rather than a minimum which is the requirement in the passivity based control techniques. In [10] a smooth feedback law is proposed whose basin of attraction is an open dense set in the state space. The set of initial conditions which does not converge to the upright position is a set of measure zero. The main idea is to design a passivity based feedback law which smoothly switches between a positive and negative feedback making use of a suitable switching function. The control law is determined as

$$(17) \quad u = \dot{\theta} \cos \theta [b^-(x) - b^+(x)],$$

$$b^-(x) = b\{d(x, L_\varepsilon(\pm \pi)) \cup B_\varepsilon(\pm x_s) - L_\varepsilon(0)\},$$

$$b^+(x) = [1 - b\{d(x, L_\varepsilon(\pm \pi))\}]b\{d(x, \{\pm x_s\}) - \varepsilon\},$$

where $L_\varepsilon(\cdot)$ and $B_\varepsilon(\cdot)$ are ε – neighborhoods around the equilibrium and saddle points of the system. Another approach for stabilization of the inverted pendulum in the upright equilibrium point is considered in [12]. Since the energy function has several minimum points, a pumping and damping strategy is needed to carry the system into the desired position. When the pendulum hangs straight down, the control signal increases the mechanical energy and thus forces the pendulum to leave this state. If the pendulum is in the upright position, the control signal decreases the mechanical energy and thus facilitates the pendulum to stay in this position. The resultant control law is continuously differentiable and entirely avoids commutation between different laws

$$(18) \quad u = 2a \sin \theta + b \dot{\theta} F(\theta, \dot{\theta}) \cos \theta,$$

where a is a parameter which is chosen to reach the maximum potential energy of the pendulum $V_p(\theta) = \cos \theta - a \cos^2 \theta - \frac{1}{4a}$.

The sizes of the pumping or damping regions are adjusted by the parameter a and the amount of damping by the parameter b . The function $F(\theta, \dot{\theta})$ is presented as follows:

$$(19) \quad F(\theta, \dot{\theta}) = \frac{\dot{\theta}^2}{2} + \frac{2a+1}{4a} (2a \cos \theta - 1).$$

It determines the regions of pumping or damping energy into the system. In [22] a feedforward control method is presented, where the nominal state trajectories for the swing up are obtained by solving an optimization problem with the stationary downward and upward equilibrium as boundary conditions. The nonlinear feedforward control is determined by solving a two-point boundary value problem for the internal dynamics of the pendulum. The control law is determined as a second derivative of the pendulum desired trajectory $u^*(t) = \ddot{y}^*(t)$. In order to determine the angular deviations of the double inverted pendulum, the following equation has to be solved

$$(20) \quad \ddot{\phi}^* = \beta(\phi^*, \dot{\phi}^*, \dot{y}^*),$$

where the input signal is replaced by the desired acceleration. The desired pendulum output trajectory which satisfies certain boundary conditions is built by using power series expansion

$$(21) \quad y^*(t) = a_0 + a_1 \cos\left(\frac{\pi t}{T}\right) + \dots + a_n \cos\left(\frac{n\pi t}{T}\right),$$

where T is the time for positioning the pendulum in the upright equilibrium state. The problem for stabilizing the inverted pendulum with a nonlinear optimal regulator by integrating the Euler – Lagrange equations backward from the origin is solved in [23]. The direct approach to obtain the nonlinear optimal feedback law $u(x)$ is to solve the optimal control problem for a large number of initial conditions $x(t_0)$ and to interpolate $u(x)$ from the vector field. By integrating the Euler – Lagrange equations backward in time, the two point boundary condition problem is replaced by the standard initial value problem. The problem of backward integration is solved in two stages: *i*) covering the state space with all possible state trajectories and *ii*) choosing this trajectory which optimizes the cost function. By solving the inverted pendulum control problem the following control law is obtained

$$(22) \quad u(\theta, \dot{\theta}) = k \left(\frac{1}{2} \dot{\theta} - 2 \sin^2 \frac{\theta}{2} \right) \dot{\theta} \cos \theta.$$

A new regulator which performs swing up and stabilization, simultaneously uses elements from input/output linearization, energy control and singular perturbation theory is proposed in [40]. The singularity in the horizontal position of the pendulum is resolved by introducing a saturation nonlinearity. An artificial reference angle is artificially introduced to enforce global stabilization which is based on the condition $(\theta - \theta_r) \dot{\theta} \leq 0$. This

condition ensures that the pendulum is moving towards the desired upright position. The reference signal θ_r takes values in the set $\theta_r \in \{-2\pi, 0, 2\pi\}$. The control law is obtained as follows:

$$(23) \quad u = \text{sat}(u_{us}, u_{max})$$

where the function sat denotes a saturation nonlinearity. The unsaturated component is determined as follows:

$$(24) \quad u_{us} = \frac{k_\theta (\theta - \theta_r) + k_\omega \dot{\theta} + \sin \theta}{\cos \theta}.$$

The method of singular perturbation is used for the cart – pendulum simultaneous control in [40], where the additional control signal component w is introduced:

$$(25) \quad w = \tan^{-1}(k_x x + k_v \dot{x}).$$

A constructive approach for inverted pendulum stabilization via immersion and invariance is proposed in [8]. The essence of the method consists of determining the system target dynamics which is described by a differentiable manifold over the state trajectories and gradual convergence of the motion towards this manifold. It is suggested that the state equations are given as: $\dot{x} = f(x) + g(x)u$. The constructive approach is described by the following steps: *i*) definition of the system target dynamics $\xi = \alpha(\xi)$, whose state ξ has a stable equilibrium point ξ^* *ii*) definition of the immersion condition

$$(26) \quad f[\pi(\xi)] + g[\pi(\xi)]c[\pi(\xi)] = \frac{\partial \pi(\xi)}{\partial \xi} \alpha(\xi),$$

where the control law is expressed as $u = c[\pi(\xi)] = c(x)$, *iii*) construction of an invariant manifold containing the motion trajectories which are described by equivalence sets: $\{x \in R^n / \phi(x) = 0\} = \{x \in R^n / x = \pi(\xi), \xi \in R^n\}$, *iv*) design of a control law in the form $u = c(x) = \psi(x, z)$, where

$$\dot{z} = \frac{\partial \phi}{\partial x} [f(x) + g(x)\psi(x, z)], \text{ that renders the manifold}$$

attractive and ensures that all signals are bounded and converge to the manifold. After applying the method of immersion and invariance to the inverted pendulum, the following control law is obtained

$$(27) \quad u = -\frac{1}{1-k_2 b} \left[\gamma k_1 + k_1 x_2 + \gamma x_3 + \frac{\gamma k_2}{\cos x_1} x_2 + k_2 \operatorname{tg} x_1 \left(\frac{x_2}{\cos x_1} + a \sin x_1 \right) \right],$$

where $k_1, k_2, k_3, a, b, \gamma$ are constants and the state equation is given by: $\dot{x}_1 = x_2$, $\dot{x}_2 = f_2(x_1) + g_2(x_1)u$, $\dot{x}_3 = f_3(x_1) + u$. The problem of positioning the inverted pendulum is solved in [46] by using a hybrid regulator. The main part of the control algorithm defines several classes of bang – bang controllers and a switching strategy which guarantee that the trajectory of the system starting from any initial condition enters in finite times the neighborhood where the locally stabilizing controller is effective. The methods of modern robust control theory can be used

if there is an uncertainty in the inverted pendulum model [4,35,42,45]. An H_∞ control algorithm is used in [42] to obtain a loop shaping regulator. The robust stabilization achieved by this regulator is based on a normalized coprime fractional representation for describing the uncertainty in the system. The structure of the regulator and the exact state space design formulas are shown. The μ – synthesis of a robust control system for the triple inverted pendulum is proposed in [4,35]. Special attention is paid on the uncertainty model which consists of two complex uncertainties in the actuators, three real uncertainties in the moments of inertia and three real uncertainties in the viscous friction coefficients. The fundamental limitations in the inverted pendulum control problem are presented in [44]. It is shown that these limitations are due to the presence of the pendulum model unstable pole. The solution of the inverted pendulum control problem by using the approach of linear matrix inequalities is proposed in [45]. The presented solution uses linear quadratic and H_∞ regulators, where the requirements for stability and nominal performance are transformed into linear matrix inequalities.

For solving the second main problem – local stabilization of the inverted pendulum in upright equilibrium position, usually linear methods are applied using the linearized pendulum model. The most popular of these methods are PID control, linear quadratic regulator and state feedback control. State feedback control is used in [27], where change of state variables is carried in advance in order to give account of the slow and fast dynamics of the system. The control algorithm generates a state feedback with simultaneously large and small gains. A position state feedback is applied in [13], where the differential component in the regulator is replaced by a pure time delay element. The control law is obtained as: $u(t) = ax(t) + bx(t - \tau)$, where a and b are tuning parameters. This is the way to avoid the adverse effects of the differential component in digital realizations. A feedback linearization approach with nonlinear observer for inverted pendulum control is proposed in [31]. A state feedback control law in Brunovski canonical form is obtained, where the states are estimated by the nonlinear observer. In order to stabilize a double inverted pendulum in upright equilibrium state, some adaptive control schemes have been used in [6,41,32]. The convergence properties of an adaptive scheme is discussed in [6,41], where it is shown how the adaptive control algorithm attends the desired performance conditions. Some control algorithms for stabilizing the inverted pendulum in downward equilibrium position are presented in [5,7,9]. These control algorithms have practical applications for a smooth crane load control. The goal here is to restrict in maximum extent the oscillations of the crane load during cart motion. This problem is solved in [7] by using a programmable logic controller which calculates the control law on the basis of velocity and current feedback. A new regulator which uses the mathematical model of the cart – crane system is obtained in [5], where a feedback correction of the angular deviation is implemented. Some special criteria for performance evaluation are introduced. The influence of the system nonlinearities is shown in [9], where a method for the nonlinearities compensation is proposed.

5. Numerical Example

Consider the inverted pendulum laboratory model shown on figure 1. The equations of dynamics for this model are presented in (1) and (2). In order to obtain the model parameter values we perform several laboratory experiments [25,34]. For example, to derive the viscous friction coefficients we determine first the pendulum damping coefficient. For this purpose we fix the cart at the left side of the rail using the spring holder and deviate the pendulum about $\theta \approx 20^\circ$ angle with respect to its lower equilibrium position. Then we let the pendulum to swing freely while gathering data from its motion. The collected data contains the oscillation amplitudes which decrease in time to zero due to the torque friction. The damping coefficient is determined from the formula [25]

$$(28) \quad \xi = \frac{2}{(2n-1)T} \lg \left(\frac{A_1}{A_n} \right),$$

where n is a number of local maximums to the left of the equilibrium position, T is the period of oscillations, A_1 and A_n are the local maximum values after 1 and n oscillations correspondingly. The viscous friction coefficient is computed by the formula

$$(29) \quad f_p = \xi \cdot J$$

where J is the pendulum moment of inertia. After performing five experiments for deviation of the pendulum to the left and to the right from the equilibrium position and removing the most distant measurements we obtain the following experimental values for the oscillation period and damping coefficient.

For deviation in left direction with respect to the lower equilibrium position:

$$T = 1.159s \quad T = 1.158s \quad T = 1.158s \\ \xi = 0.033s^{-1} \quad \xi = 0.0345s^{-1} \quad \xi = 0.034s^{-1}$$

For deviation in right direction with respect to the lower equilibrium position:

$$T = 1.157s \quad T = 1.157s \quad T = 1.158s \\ \xi = 0.0436s^{-1} \quad \xi = 0.0399s^{-1} \quad \xi = 0.0347s^{-1}$$

After averaging the gathered data we obtain the final values:

$$T = 1.158s \quad \xi = 0.0366s^{-1}$$

The following viscous friction coefficient for rotational motion is calculated

$$f_p = \xi \cdot J = 0.0366 \cdot 2.839 \cdot 10^{-3} = 0.1035 \cdot 10^{-3} \frac{kg \cdot m^2}{sec}.$$

The other model parameter values are obtained as follows [34]:

$$M = 0.768kg, m = 0.104kg, l = 0.1735m,$$

$$J = 2.839 \cdot 10^{-3} kgm^2$$

$$F = \frac{G}{R} \frac{k_i}{R_a} u - \frac{G^2}{R^2} \frac{k_b k_i}{R_a} \dot{x} = p_1 u + p_2 \dot{x}, \quad \text{where } p_1 = 11.81 \quad \text{and}$$

$$p_2 = -0.865 \frac{Ns}{m}$$

The estimated model is used for designing the pendulum controllers. Two control tasks are performed: swing up control and stabilization control. Swing up control is based on the energy algorithm and stabilization control utilizes the PID algorithm. The following steps describe the controller operation:

1. Check if the pendulum is in the stabilization region

$$\text{if } |\theta| - \Delta\theta < 0$$

2. Start the local linear regulator

$$u \rightarrow \text{PID}(\varepsilon)$$

else

3. Check if the kinetic and potential energies are large enough to overcome the gravity and swinging up the pendulum

$$\text{if } \left\{ \frac{1}{2}\dot{\theta}^2 + gml[\cos(\theta) - 1] \right\} > 0$$

4. Set the control variable to zero

$$u = 0$$

else

5. Start the algorithm for swinging up the pendulum

$$u = u_{\max} \text{sign}[\dot{\theta}(|\theta| - \Delta\theta)]$$

end

end

6. Overcome the effect of the Coulomb friction force

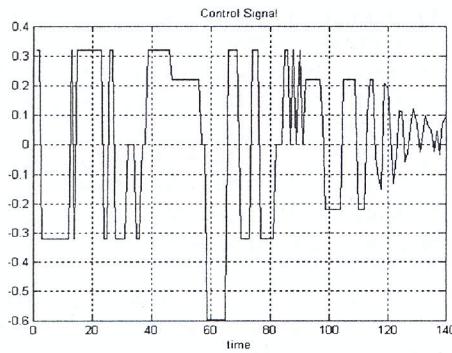
if $u > 0$

$$u = u + F_c$$

else

$$u = u - F_c$$

end



in state space as

$$(31) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J+ml^2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-ml}{J+ml^2} \end{bmatrix} u,$$

where $x_1 = \theta$ and $x_2 = \dot{\theta}$. If we choose a sampling time of $T_0 = 0.25$ sec, we obtain a discretized model

$$x_{k+1} = Ax_k + Bu_k \text{ with matrices } A_d = \begin{bmatrix} 1.673 & 0.3039 \\ 5.917 & 1.673 \end{bmatrix}$$

and $B_d = \begin{bmatrix} -0.069 \\ -0.6 \end{bmatrix}$. The cost function of the predictive controller can be written as [2]

$$(32) \quad J = \sum_{i=1}^N (r_{k+i} - y_{k+i})^2 + \lambda \sum_{i=0}^{N-1} (\Delta u_{k+i})^2,$$

where the extended model for predictive control has been used

$$(33) \quad \begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} B \\ 1 \end{bmatrix} \Delta u_k;$$

$$(34) \quad y_k = \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} + D \Delta u_k + d_k,$$

d_k is a disturbance at the output and $D = 0$. The control law is computed from the optimality condition on the cost function

$$\frac{dJ}{d \Delta u} = 0. \text{ Finally we obtain [2]}$$

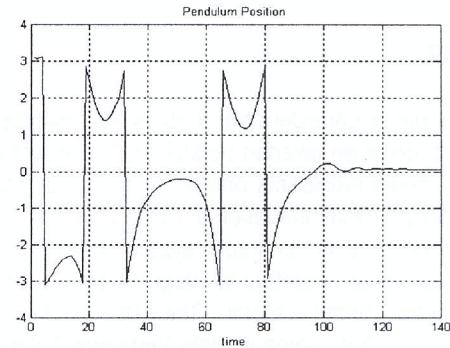


Figure 3. The control signal and the angular deviation of the inverted pendulum

Experimental results from the application of the algorithm are shown in figure 3 [3]:

The stabilization of the pendulum in upright equilibrium state can be implemented by using the model based predictive control method [2]. For this purpose we use a simplified model of the pendulum linearized in the upright equilibrium position:

$$(30) \quad (J + ml^2)\ddot{\theta} = mlg\theta - mlu,$$

where it is assumed that the linear acceleration u determines the control signal to the pendulum. Equation (30) can be presented

$$(35) \quad \Delta u_k = e_i^T (H^T H + \lambda I)^{-1} H^T \left(r - \begin{bmatrix} P & L \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ d_k \end{bmatrix} \right),$$

$$\text{where } P = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^N \end{bmatrix}, \quad H = \begin{bmatrix} CB & 0 & 0 & \dots \\ CAB & CB & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ CA^{N-1}B & CA^{N-2}B & CA^{N-3}B & \dots \end{bmatrix}, \quad L = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \text{ and}$$

$$\hat{x}_k = \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix}, \quad r = (r_{k+1} \ r_{k+2} \ \dots \ r_{k+N})^T \quad \text{and}$$

$$e_1 = (1 \ 0 \ \dots \ 0)^T.$$

We assume that the pendulum is in its upright unstable equilibrium position and apply an impulsive disturbance with duration of 3 sec. In figure 4 it is shown the reaction of the pendulum control system to the impulsive disturbance. It is observed that the control system performance depends on the chosen sampling period. When $T_0 = 0.05$ sec the closed-loop system is unstable, however for $T_0 = 0.01$ sec the closed-loop system is stable and attenuates the disturbance. Thus, for larger sampling times the closed-loop system becomes unstable and the pendulum departs from the equilibrium position.

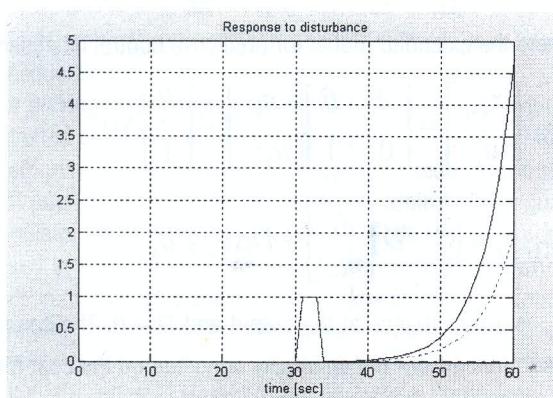


Figure 4. Pendulum disturbance reaction ----- $T_0=0.055$, $T_0=0.05$, -.-. $T_0=0.01$

6. Conclusion

This overview paper considers the problem of swing up and upright stabilization of an inverted pendulum. The inverted pendulum physical model represents one of the most popular laboratory models for practical realization of control systems in laboratory conditions. It is an unstable system with many nonlinearities and equilibrium points and degrees of freedom exceeding the number of control actuators. The cart – pendulum system serves to demonstrate some complicated control algorithms and their verification under different performance limitations. The paper presents the two main approaches for pendulum swing up control: the energy approach and the speed gradient approach. At the same time, many modern methods for control of such underactuated systems like the inverted pendulum are introduced. The presented methods solve the inverted pendulum swing up problem and ensure its global stabilization. A numerical example is presented to resemble the main stages of solving the inverted pendulum control problem.

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