Bi-level Optimization for Portfolio Modelling

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Key Words: Financial portfolio; bi-level optimization; hierarchical theory.

Abstract. The optimal resource allocation of sets of securities, available in the financial market can be done by solving a multi-criterion optimization task, aimed at maximization of portfolio return and minimization of portfolio risk. Here is proposed the utilization of hierarchical coordination for solving the bi-level optimization problem, which formalizes the investment process. A two-level hierarchical approach for solving the portfolio optimization problem is applied where the optimal Sharp ratio of risk versus return is solved at the upper level and as a result is determined by the investor's preferences for taking risk on the basis of objective considerations. The optimal portfolio is evaluated at the lower level.

Introduction

The problem of portfolio optimization targets the optimal resource allocation in the investment process [12]. The resource allocation is made by investing capital in financial assets (or goods), which give return to the investor after a period of time. For the investment process the target is to maximize the return while the investment risk has to be minimal [2,6,7,8,10,11]. The risk is equivalent to uncertainty. The term "risk" reflects the undetermined and non-predictable future. The minimization of financial risk is with high priority during the investment and that is why the statistics and probability modelling are interested in it. The financial risk is always related with the portfolio management [20]. The difficulties of predicting the financial risk are related to the market behaviour, based on continuous dynamical changes. The investment models are based on mathematical analytical tools, which formalize both the behavior of the market players and future events in financial markets. In order to formalize the investment process, financial resource allocation has to be done. This requires a market analysis, which usually uses predefined assumptions. Usually, such assumptions concern uncertainty in ideal mathematical behaviour, constant and not changing environment influences.

In portfolio theory the decision maker makes decisions taking into account the risk of the investment. The portfolio optimization models are based mainly on probability theory. However, the probabilistic approach is not able to formalize the real market behaviour. Another uncertainty modeling approach of the financial market is the fuzzy set theory [4].

An essential contribution for the finance modelling and especially for risk assessment is the work of Markowitz [9] which concerns the individual investor. This theory is based on both optimization and probability theory. The investor's goal is to maximize the return and to minimize the risk of the investment decisions. The investor's return is formalized as the mean value of a random behaved function of the portfolio securities returns. The risk is formalized as a variance of these portfolio securities. The portfolio modelling is formalized by the above mathematical representations of return and risk which define the portfolio optimization problem. The portfolio solution depends on the level of risk which investor can bear in comparison with the level of portfolio return. Thus, for the practical utilization of the portfolio theory, the relation between return and risk is the main parameter for the investor. The portfolio risk is minimized according to two types of arguments: the portfolio content and the parameter of the investor's risk preference. The market risk, which results in different values of the variances of the average return, is under consideration in the paper. The market risk is defined as a risk to the financial portfolio, related to the dynamic changes of the market prices of equity, foreign exchange rates, interest rates, commodity prices [3]. The financial firms generally take a market risk to receive profits. Particularly, they try to take a risk they intend to have and they actively manage the market risk.

Usually, the investment decision-making process is done by investors' subjective assumptions about the the relationship between portfolio risk andreturn. In this paper decreasing the subjective influence in the investment process is proposed. This is achieved by calculating the unknown investor's coefficient for undertaking risk based on the optimization problem. A bi-level optimization problem based on a hierarchical system's modelling formalizes the portfolio investment. The parameter of the investor's risk preference is evaluated at the upper level. After that, this parameter is used for optimal resource allocation of the portfolio optimization problem by minimizing risk and maximizing return. In that manner, the process of portfolio resource allocation is performed without subjective influence.

Portfolio Optimization Problem

The portfolio theory is developed to support decision making for investment allocation of financial assets selling (securities, bounds) at the stock exchange [1]. This allocation is known as "investment" decision making. The investor considers the asset as a matter of future income. The better combination of financial assets (securities) of the portfolio leads to better return for the investor. The portfolio contains a set of securities. The problem of portfolio optimization targets the optimal resource allocation in investment process of trading financial assets [12]. The resource allocation means investing capital in financial assets (or goods), which gives return to the investor after a period of time. For the investment process the target is to maximize the return while the investment risk has to be minimal [11]. Harry Markowitz suggested a powerful approach for quantifying the risk in 1952. The analytical relations among the portfolio risk V_p, portfolio return E_p and the values of the investment per type of assets x_i, according to the portfolio theory, are [12]

$$E_p = \sum_{i=1}^{n} E_i x_i = E^T x;$$

$$V_p = \sum_{j}^{n} \sum_{i=1}^{n} x_i x_j \operatorname{cov}(i,j) = x^T \operatorname{cov}(.) x ,$$

where

E_i – is the average value of the return of asset i;

$E^T = (E_1,..., E_n)^T$ – is a vector with dimension 1 x n;

cov(i,j) – is the co-variation coefficient between the assets i and j.

The component $V_p = x^T \operatorname{cov}(.)\, x$ formalizes the quantitative assessment of the portfolio risk. The component $E_p = E^T x$ is the quantitative evaluation of the portfolio return. The portfolio problem solutions x_i, $i = 1,n$ determine the relative amounts of the investment per security i.

The co-variation is calculated from previously available statistical data for the returns of assets i and j and it represents a symmetrical matrix

$$\operatorname{cov}(.) = \begin{Vmatrix} \operatorname{cov}(1,1) & \operatorname{cov}(1,2) & \cdots & \operatorname{cov}(1,n) \\ \operatorname{cov}(2,1) & \operatorname{cov}(2,2) & \cdots & \operatorname{cov}(2,n) \\ \vdots & & & \\ \operatorname{cov}(n,1) & \operatorname{cov}(n,2) & \cdots & \operatorname{cov}(n,n) \end{Vmatrix}_{nxn} .$$

The components of cov(i,j) are evaluated from $R_i^{(1)}, R_i^2, \cdots, R_i^{(N)}$ and $R_j^{(1)}, R_j^2, \cdots, R_j^{(N)}$, which concern the profit of assets i and j for discrete time moments (1), (2),…, (N). The co-variation coefficient between assets i and j is calculated like

$$\operatorname{cov}(i,j) = \frac{1}{N} \begin{bmatrix} (R_i^{(1)} - E_i)(R_j^{(1)} - E_j) + \\ (R_i^{(2)} - E_i)(R_j^{(2)} - E_j) + \\ + \cdots (R_i^{(N)} - E_i)(R_j^{(N)} - E_j) \end{bmatrix},$$

where

$$E_i = \frac{1}{N}\left[R_i^{(1)} + R_i^{(2)} + \cdots + R_i^{(N)}\right],$$

$$E_j = \frac{1}{N}\left[R_j^{(1)} + R_j^{(2)} + \cdots + R_j^{(N)}\right]$$

are the average profits of assets i and j for the period $T = [1,2,...., N]$. The portfolio theory defines the so called "standard" optimization problem [12] by:

$$(1) \quad \min_x \left[\frac{1}{2} x^T \operatorname{cov}(.)\, x - \sigma E^T x\right],$$

$$x^T.\mathbf{1} = 1 ,$$

where

cov(.) – a symmetric positively defined square matrix n x n,

E – (n x 1) vector of the average profits of the assets for the period of time T = [1,2, ... , N];

1 = |1 ... |^T – unity vector, $n \times 1$;

σ – is a parameter of the investor's preferences to undertake risk in the investment process.

The constraint is equivalent to the equation $x_1 + x_2 + ... + x_n = 1$, which formalizes the fact that the investment is not partly implemented or the full amount of the resources are devoted to the investments. If the right side of the constraint is less than 1, this means that all amounts of the investments are not effectively used. The investment per different assets has to be performed for the total amount of the available investment resources, numerically presented as relative value of 1. The solutions x_i, i=1,n give the relative values of the investment, which are allocated for the assets i, i=1,n.

The component of the goal function $V_p = x^T \operatorname{cov}(.)\, x$ is the quantitative assessment of the portfolio risk. The component $E_p = E^T x$ is the quantitative value of the portfolio return. The goal function of problem (1) targets the minimization of the portfolio risk V_p and maximization of its return E_p. The parameter σ belongs to the range [0, + ∞] and formalizes the investor's tendency to undertake risk. For $\sigma = 0$ the investor is very cautious (even cowardly) and his general task is to decrease the risk of the investment, $\min_x [x^T \operatorname{cov}(.) x]$. For $\sigma = +\infty$ the investor is far-away from the risk in the investments. His target is to obtain a maximum return from the investment. For that case the relative weight of the return in the goal function is most weighted, and then the optimization problem has an analytical form: $\min_x[-\sigma E^T x] \equiv \max_x [E^T x]$.

Thus, in the portfolio problem the unknown parameter σ is presented, which assesses the investor's preferences for undertaking risk in decision making. This parameter influences the portfolio problem, making it a parametric one. Respectively, for a new value of σ, the portfolio problem (1) has to be solved again. In the trivial case when σ is not properly estimated, the optimization problem has to be solved for a set of σ. The values σ introduce strong subjective influence to the solutions of the portfolio problem. Additionally, for practical reasons, the portfolio problem has to be solved multiple times with a set of values for the coefficient of the investor's preferences σ to undertake risk. The complexity of solution of (1) and estimation of σ is an obstacle for real-time investment applications.

The numerical assessment of σ is a task of the financial analyzer and it has subjective meaning. This coefficient strongly influences the definition and respectively the solutions of the portfolio problem. Respectively, σ changes the final investment decision as well.

The assessment of the portfolio characteristics is found as combinations of admissible assets in the space risk-return $V_p=V_p(E_p)$. The investor has to choose the optimal portfolio from the upper set of admissible solutions named "efficiency frontier". The "efficiency frontier" is found with difficulty and slowly. Each point from this curve is found after solving the portfolio optimization problem with different values of σ. The "efficient frontier" is evaluated point after point according to iterative numerical procedure:

1. Choice of initial value of σ for the investor's preferences. A good starting point is $\sigma = 0$. It corresponds to the case of investor who is not keen in risky decisions;
2. The portfolio problem is solved with the chosen σ

$$\min_x [\frac{1}{2} x^T Q x + R^T x]$$

$$x^T \times 1 = 1$$

and the optimal solution $x(\sigma)$ is found.

3. Evaluation of portfolio risk and portfolio return:

V_p - $x^T(\sigma)\mathrm{cov}(.)x^T(\sigma)$, $E_p = E^T x(\sigma)$.

These values give a point into the space $V_p = V_p(E_p)$, which belongs to the efficient frontier;

4. New value $\sigma_{new}=\sigma_{old}+\Delta$ is chosen, where Δ is determined by considerations for completeness in moving into the set s = $[0,+\infty]$. Jump to point 2.

Hence, for each solution of the portfolio optimization problem one point of space $V_p = V_p(E_p)$, belonging to the curve of the efficient frontier is found, *figure 1*. Problem (1) is solved with number of values of σ. A set of solutions $x(\sigma)$ is found while the best value of σ^* for that investor is empirically estimated and the optimal portfolio solution $x(\sigma^*)$ is determined.

Such an approach, however, causes a contradiction between the manner of quantitative definition of problem (1) and the final decision for the investment. According to the portfolio theory the value of σ^* has to be determined before solution of the problem. However, σ^* is estimated after

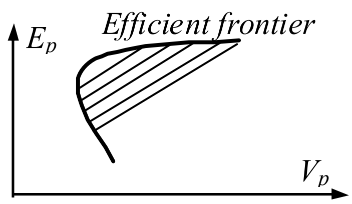

Figure 1. Efficient frontier of the portfolio optimization

evaluation of number of portfolio problems (1) with different values of σ, determined by the investor. In other words, evaluation of σ^* is by subjective manner.

The model, developed here, ignores the subjective influence during evaluation and assessment of the parameter of investor's preference to risk σ. The decision making process is formalized by two hierarchically interconnected optimization problems, *figure 2*. σ is determined by solution of optimization problem at the upper hierarchical level. This problem is defined without subjective considerations. For example, this optimization problem can aim evaluating of σ, which leads to a good ratio between the portfolio risk and

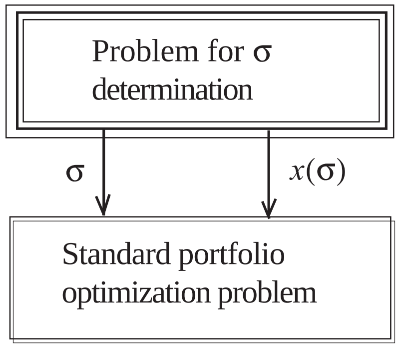

Figure 2. Bi-level portfolio optimization problem

return.

At the lower hierarchical level the standard portfolio optimization problem is solved using σ, estimated from the optimization problem at the upper level.

Hierarchical Bi-level Optimization Problem

Typical for the bi-level optimization problems is that the solution to an optimization problem at the upper influence the lower level optimization problems. The lower level solutions define on their turn a set of parameters for the upper level problem. Because of the interrelation between the upper and lower level optimization problems by their solutions, the exact form of the optimization problems cannot be defined. This complexity of the related bi-level optimization problems is solved by numerical iterative calculations of both problems (at the lower and upper hierarchical levels) until reaching the optimal solutions.

The general bi-level hierarchical optimization problem is based on the formulation of the Stackelberg game [16]. The Stackelberg problem can be interpreted as a game between two players, each of them making decisions [13,14,15]. The decisions of the leader (upper level problem) answer the questions: which is the best strategy for the leader, if he knows the goal function and the constraints of the follower (lower level problem) and how the leader has to choose his next decisions? When the leader evaluates his decisions, the follower chooses his own strategy for decision making for minimization of his goal function. Respectively, the follower solves the appropriate optimization problem. Usually, both problems are nonlinear optimization problems from mathematical programming. Due to methodological difficulties for the solution of hierarchically interconnected optimization problems, today the classical application of the portfolio theory lacks in solving bi-level optimization problems. Currently, the portfolio problem is solved by quantitative assessment of σ^* in advance, without applying interconnected hierarchical optimization. The value of σ^* is estimated intuitively or empirically by an expert. Here, for the solution of the bi-level portfolio problem we apply a methodology, derived as non-iterative coordination [17,19]. The methodology for non-iterative coordination in hierarchical systems defines analytical approximations of the inexplicit function, used by the upper and lower optimization problems. In that manner, analytical relations between the investor's preferences for the risk σ and the solutions x_i are derived [18]. Such relations support fast solution of the bi-level problem and respectively allow real time decision making. The upper level problem is defined with a goal function, which minimizes the Sharp ratio: portfolio risk versus portfolio return. The argument of this optimization problem is the investor's preferences for taking risk σ. Applying the non-iterative methodology [17,19], analytical relations between the portfolio problem's parameters E_p, V_p , the portfolio solutions x_i and the parameter of the investor's preference σ are derived. These relations allow speeding up the decision making process and the investment decisions can be done in real time.

Portfolio Bi-level Problem's Solution

The initial problem (1) has solutions x_i , which have to be described as analytical functions of σ parameter. For that case the initial problem (1) is rewritten in the form

$$(2)\quad \min_x [\frac{1}{2} x^T Qx + R^T x]$$

$$Ax = C,$$

where the correspondence between (1) and (2) is:

Q=cov(.), $R=-\sigma E$, A=1, C=1.

If σ is known, problem (1) has a solution, denoted like $x(\sigma)$. For the case when varies, the solution of the portfolio problem x is an inexplicit analytical function of σ, or $x=x(\sigma)$.

Respectively, the portfolio risk

$V_p(\sigma) = x^T(\sigma)\text{cov}(.)x(\sigma)$

and the portfolio return

$E_p(\sigma) = E^T x(\sigma)$

are also inexplicit functions of σ.

Problem (2) can be solved, applying the method of the non-iterative coordination, which gives possibility to be derived approximations of the inexplicit analytical relations of the portfolio parameters $V_p(\sigma)$, $E_p(\sigma)$, $x(\sigma)$ towards the argument σ. Using results from [19], the analytical solution of problem (2) is

$$(3)\quad x^{opt} = -Q^{-1}[R - A^T(AQ^{-1}A^T)^{-1}(AQ^{-1}R + C)] .$$

Using this relation, the analytical descriptions of the portfolio risk and return become

$$V_p = x^{Topt} Q x^{opt} =$$
$$= \left\{-\left[(C^T + R^T Q^{-1} A^T)(-AQ^{-1}A^T)^{-1} A + R^T\right] Q^{-1}\right\} Q .$$
$$\left\{-Q^{-1}\left[R - A^T(AQ^{-1}A^T)^{-1}(AQ^{-1}R + C)\right]\right\}$$

After several transformations it follows

$$V_p = R^T Q^{-1}\left[R - A^T(AQ^{-1}A^T)^{-1} A\right] Q^{-1} R +$$
$$+ C^T(AQ^{-1}A^T)^{-1} C \qquad .$$

The analytical relation of the portfolio return is obtained as a linear relation towards x_{opt} or

$$E_p = E^T x = R^T x^{opt} = R^T\left\{-Q^{-1}\left[R - A^T(AQ^{-1}A^T)^{-1}(AQ^{-1}R + C)\right]\right\} =$$
$$= -R^T Q^{-1}[R - A^T(AQ^{-1}A^T)^{-1} AQ^{-1}R] + R^T Q^{-1} A^T (AQ^{-1}A^T)^{-1} C \qquad .$$

Finally, it holds

$$(4)\quad V_p = R^T Q^{-1}\left[R - A^T(AQ^{-1}A^T)^{-1} A\right] Q^{-1} R + C^T(AQ^{-1}A^T)^{-1} C$$

$$(5)\quad E_p = -R^T Q^{-1}[R - A^T(AQ^{-1}A^T)^{-1} AQ^{-1}R] +$$
$$+ R^T Q^{-1} A^T (AQ^{-1}A^T)^{-1} \mathrm{C}.$$

Relations (4) and (5) can be expressed in terms of the initial portfolio problem (1). Thus, explicit analytical relations for the portfolio risk V_p, portfolio return E_p and the optimal solution of the portfolio problem x_{opt} are derived towards the coefficient of the investor's risk preference σ. For the current problem (1), taking into account the correspondence between problems (1) and (2), it follows

$$(6)\quad x^{opt}(\sigma) = Q^{-1}\{\left[E - A^T(AQ^{-1}A^T)^{-1} AQ^{-1}E\right]\sigma +$$
$$+ A^T(AQ^{-1}A^T)^{-1} C\}$$

$$(7)\quad V_p(\sigma) = E^T Q^{-1}[E - A^T(AQ^{-1}A^T)^{-1} AQ^{-1}E]\sigma^2 + C^T(AQ^{-1}A^T)^{-1} C$$

$$(8)\quad E_p(\sigma) = E^T x^{opt}(\sigma) = E^T Q^{-1}\{\left[E - A^T(AQ^{-1}A^T)^{-1} AQ^{-1}E\right]\sigma + A^T(AQ^{-1}A^T)^{-1} C\}.$$

The following notations are used to simplify the expressions

$$(9)\quad \alpha = E^T Q^{-1}\left[E - A^T(AQ^{-1}A^T)^{-1} AQ^{-1}E\right]$$
$$\beta = C^T(AQ^{-1}A^T)^{-1} C$$
$$\gamma = E^T Q^{-1} A^T (AQ^{-1}A^T)^{-1} C ,$$

where the parameters α, β and γ are scalars. Respectively, relations (7) and (8) become

$$V_p(\sigma) = \alpha\sigma^2 + \beta,$$
$$(10)\quad E_p(\sigma) = \alpha\sigma + \gamma.$$

The derived relations (6), (7), (8) and (9) describe in analytically explicit form the functional relations between the portfolio parameters for risk, return and optimal solution towards the coefficient of the investor's preferences to risk σ. Hence, the solution of the portfolio problem (1) is calculated using relations (6)-(8) without implementation of optimization algorithms for the solution of the low level optimization problem. This considerably speeds up the problem solution of (1). Hence, the portfolio optimization problem can be solved in real time, with lack of iterative calculations, which benefits the decision making in fast dynamical environment of stock exchange.

On the upper optimization level it is necessary to be evaluated σ – the parameter of investor's preferences, under which the better (minimal) value of the relation Risk/Return for the optimal portfolio is achieved. This relation is known as Sharp ratio. The problem for the evaluation of σ in formal way is stated like

$$\min_{\sigma \geq 0} \left\{\frac{Risk(\sigma)}{Portfolio\_return(\sigma)} = \frac{V_p(\sigma)}{E_p(\sigma)}\right\}.$$

According to relation (10) the analytical form of the problem is

$$(11)\quad \min_{\sigma}\left\{\frac{V_p(\sigma)}{E_p(\sigma)} = \frac{\alpha\sigma^2 + \beta}{\alpha\sigma + \gamma} = \eta(\sigma)\right\}.$$

This problem evaluates the parameter of the investor's preferences σ according to objective considerations. Thus, the portfolio optimization problem is stated like a bi-level optimization procedure, *figure 2*. The advantage for the evaluation of σ comes from the fact that the estimation of σ is done overcoming the subjective influences of the investor, and it is found from a real optimization problem.

The solution σ_{opt} of such a problem is found according to the relations

$$\sigma_{opt} \equiv \arg\left\{ \min\left[0, \quad \frac{d\eta(\sigma)}{d\sigma} = 0\right]\right\}$$

or

$$\sigma_{opt} = \min\left[0, \quad \frac{d}{d\sigma}\left(\frac{\alpha\sigma^2+\beta}{\alpha\sigma+\gamma}\right) = 0\right]$$

$$\frac{d\eta(\sigma)}{d\sigma} = \frac{2\alpha\sigma(\alpha\sigma+\gamma)-\alpha(\alpha\sigma^2+\beta)}{(\alpha\sigma+\gamma)^2} = 0.$$

It is necessary to satisfy the condition

$$(12) \quad \alpha\sigma+\gamma = E_p > 0, \qquad \alpha \neq 0.$$

Respectively, it holds

$$\sigma_1^{opt} = \frac{-\gamma+\sqrt{\gamma^2+\alpha\beta}}{\alpha} = \frac{-1+\sqrt{1+\frac{\alpha}{\gamma}\frac{\beta}{\gamma}}}{\frac{\alpha}{\gamma}}.$$

For the particular case when C is a digital number (C=1 for relative assessment of the investment), it holds

$$(13) \quad \sigma^{opt} = \min\left[0, \quad \frac{\gamma}{2}\left(-1+\sqrt{\frac{E^TQ^{-1}E}{E^TQ^{-1}A^T(AQ^{-1}A^T)^{-1}AQ^{-1}E}}\right)\right].$$

This relation gives analytical way of calculation of the optimal parameter for risk preferences of the investor. For that reason the solution of the upper level optimization problem is reduced to analytical relation (13), applied for the calculation of σ_{opt}.

Bi-level Optimization Example

An illustration of the solution of a set of bi-level optimization problems is given below. A set of optimization problems is defined with a maximal amount of 21 securities, traded at the Bulgarian stock exchange, n=21 for 12 months (2011). The portfolio optimization problem has been defined and solved with 21 securities by two manners: using MATLAB – function QP and by non-iterative coordination. The matrices for the portfolio problem $Q|_{21x21}$ and $E|_{21x1}$ are defined according to the Bulgarian stock's exchange data as follows:

Q_{21}=[0.0540 0.0205 0.0988 0.0970 0.1136 0.0065 0.0734 0.0906 0.1440 0.1224
0.1196 -0.3193 0.3684 0.1213 0.1943 0.1576 0.1296 0.0705 0.0463 0.0371 0.2702;

0.0205 0.1116 0.0983 0.0454 0.0886 0.0056 0.0505 0.1002 0.0558 0.1306
0.1207 -0.1589 0.0702 0.0231 0.0209 0.1435 0.1211 0.8968 0.0200 0.0111 0.0906;

0.0988 0.0983 0.3913 0.2086 0.2722 0.0150 0.1706 0.2424 0.3216 0.3213
0.2084 -0.8076 0.8072 0.2308 0.3890 0.3982 0.3248 0.7874 0.1028 0.0802 0.5857;

0.0970 0.0454 0.2086 0.2407 0.2465 0.0141 0.1643 0.2019 0.3105 0.2679
0.2724 -0.6105 0.8744 0.2304 0.4407 0.3469 0.2816 -0.0198 0.1033 0.0861 0.5885;

0.1136 0.0886 0.2722 0.2465 0.3209 0.0184 0.1968 0.2550 0.3662 0.3450
0.3322 -0.8360 0.8724 0.3035 0.4750 0.4347 0.3622 0.6320 0.1181 0.0918 0.6871;

0.0065 0.0056 0.0150 0.0141 0.0184 0.0112 0.0117 0.0149 0.0203 0.0204
0.0208 -0.0465 0.0479 0.0167 0.0269 0.0257 0.0218 0.0609 0.0066 0.0051 0.0393;

0.0734 0.0505 0.1706 0.1643 0.1968 0.0117 0.1374 0.1598 0.2315 0.2180
0.2162 -0.5025 0.5998 0.1952 0.3087 0.2773 0.2304 0.3660 0.0758 0.0603 0.4417;

0.0906 0.1002 0.2424 0.2019 0.2550 0.0149 0.1598 0.2429 0.3029 0.2974
0.2908 -0.7189 0.7113 0.1681 0.3790 0.3669 0.2983 0.5178 0.0984 0.0780 0.5420;

0.1440 0.0558 0.3216 0.3105 0.3662 0.0203 0.2315 0.3029 0.5194 0.3875
0.3746 -1.2041 1.1911 0.3475 0.6786 0.4987 0.4008 -0.3238 0.1601 0.1286 0.9098;

0.1224 0.1306 0.3213 0.2679 0.3450 0.0204 0.2180 0.2974 0.3875 0.4116
0.3896 -0.9008 0.9527 0.3177 0.4763 0.4973 0.4125 1.0202 0.1262 0.0970 0.7210;

0.1196 0.1207 0.2084 0.2724 0.3322 0.0208 0.2162 0.2908 0.3746 0.3896
0.5162 -0.8601 1.0153 0.2728 0.4710 0.4859 0.4003 1.0033 0.1243 0.1000 0.7031;

-0.3193 -0.1589 -0.8076 -0.6105 -0.8360 -0.0465 -0.5025 -0.7189 -1.2041 -0.9008
-0.8601 3.3829 -2.4404 -0.7374 -1.4687 -1.1375 -0.9112 -0.0726 -0.3619 -0.2811 -2.0580;

0.3684 0.0702 0.8072 0.8744 0.8724 0.0479 0.5998 0.7113 1.1911 0.9527
1.0153 -2.4404 3.7965 0.9576 1.6834 1.2574 0.9953 -0.9148 0.3931 0.3366 2.2460;

0.1213 0.0231 0.2308 0.2304 0.3035 0.0167 0.1952 0.1681 0.3475 0.3177
0.2728 -0.7374 0.9576 0.8309 0.3990 0.4141 0.3651 0.6550 0.1063 0.0654 0.7164;

0.1943 0.0209 0.3890 0.4407 0.4750 0.0269 0.3087 0.3790 0.6786 0.4763
0.4710 -1.4687 1.6834 0.3990 1.0035 0.6317 0.5072 -0.9903 0.2173 0.1832 1.2448;

0.1576 0.1435 0.3982 0.3469 0.4347 0.0257 0.2773 0.3669 0.4987 0.4973
0.4859 -1.1375 1.2574 0.4141 0.6317 0.6321 0.5159 1.1086 0.1625 0.1267 0.9361;

0.1296 0.1211 0.3248 0.2816 0.3622 0.0218 0.2304 0.2983 0.4008 0.4125
0.4003 -0.9112 0.9953 0.3651 0.5072 0.5159 0.4434 1.1399 0.1307 0.1003 0.7665;

0.0705 0.8968 0.7874 -0.0198 0.6320 0.0609 0.3660 0.5178 -0.3238 1.0202
1.0033 -0.0726 -0.9148 0.6550 -0.9903 1.1086 1.1399 20.3171 -0.0831 -0.1369 -0.0672;

0.0463 0.0200 0.1028 0.1033 0.1181 0.0066 0.0758 0.0984 0.1601 0.1262
0.1243 -0.3619 0.3931 0.1063 0.2173 0.1625 0.1307 -0.0831 0.0612 0.0417 0.2896;

0.0371 0.0111 0.0802 0.0861 0.0918 0.0051 0.0603 0.0780 0.1286 0.0970
0.1000 -0.2811 0.3366 0.0654 0.1832 0.1267 0.1003 -0.1369 0.0417 0.0456 0.2333;

0.2702 0.0906 0.5857 0.5885 0.6871 0.0393 0.4417 0.5420 0.9098 0.7210
0.7031 -2.0580 2.2460 0.7164 1.2448 0.9361 0.7665 -0.0672 0.2896 0.2333 1.6988]

E_{21}^T =[0.892; 6.2288; 3.2956; 1.2593; 3.8264; 1.3893; 1.2487; 2.6720; 2.6345; 2.7124; 6.4636; 87.4296; 8.4614; 4.7423; 5.7003; 2.5624; 2.7413; 77.3678; 0.5254; 1.2075; 5.5153];

The target of the experiments is to evaluate the efficient frontier for the optimization problem where sigma varies from 0.001 to different upper levels (10, 30, 90) with increment of 0.1. Then, having the efficient frontier, the optimization procedure continues with finding that portfolio, which has minimal Sharp ratio (risk versus return). For that case the parameter of the investor's preferences for risk σ_{opt} is calculated, using relation (13).

The sequence of the solution of the portfolio problem is the following:

1. Analytical definition of the portfolio problem (1) with n=21;
2. Evaluation of the scalar values of the intermediate parameters $\alpha(n)$, $\beta(n)$, $\gamma(n)$ from (9);
3. Starting the calculations of the efficient frontier with initial value $\sigma^*=0$;
4. Evaluation of the portfolio parameters $V_p=V_p(\sigma^*, \alpha(n), \beta(n), \gamma(n)$, $E_p=E_p(s^*, \alpha(n), \beta(n), \gamma(n))$, according to (10). Thus, one point from the efficient frontier in the space risk/return $V_p^*\,(E_p^*)$ is found;
5. New value of the coefficient σ is chosen, $\sigma^{**}=\sigma^*+0.1$. Jump to 4.

These steps are performed for different points of the graphics $V_p = V_p\,(E_p)$ depending on the upper bound of σ. Relation Return – σ when σ varies with 0.1 from 0.001 to 30 and 90 is presented in *figure 3* and *figure 4,* respectively.

Relation Risk – σ when σ varies with 0.1 from 0.001 to 30 and 90 is presented respectively in *figure 5* and *figure.6*.

The efficient frontier $V_p=V_p(E_p)$ in the classical case is determined point after point after numerous solutions of the portfolio optimization problem with different values of the parameter of the investor's preferences to risk σ. After drawing the efficient frontier, the optimal value of σ is determined. In this paper following problem (11), it has been calculated σ_{opt}, as a solution of an upper level optimization problem

$$\min_{\sigma}\left\{\frac{Risk(\sigma)}{\mathrm{Re}\,turn(\sigma)}\right\},$$

$$(14)\qquad \min_{\sigma}\left\{\frac{V_p(\sigma)}{E_p(\sigma)}=\frac{x^T(\sigma)Qx(\sigma)}{E^T x(\sigma)}\right\},$$

where $x(\sigma)$ is an inexplicit function, defined by the solution of the portfolio optimization problem (1) for different values of σ, *figure 7* and *figure 8* . Problem (14) introduces an objective criterion for the choice and estimation of the coefficient of the investor's preferences. While in the classical model of the portfolio optimization σ is chosen by the financial analyzer, here $s\sigma$ is determined by solving optimization problem (14). The optimal value of σ_{opt} is calculated using relation (13) and its numerical result here is 0.00016926. It is seen on the graphics Risk-Return as a pink star – near to 0, *figure 7* and *figure 8*.

Figures 9 and *10* present the relation for different upper values of σ - 30 and 90, respectively. These graphics explicitly demonstrate the minimum towards σ of the ratio of portfolio risk versus return. The corresponding value σ_{opt} is found according to objective considerations, coming from the upper level optimization problem for minimization of Sharp ratio according to (15)

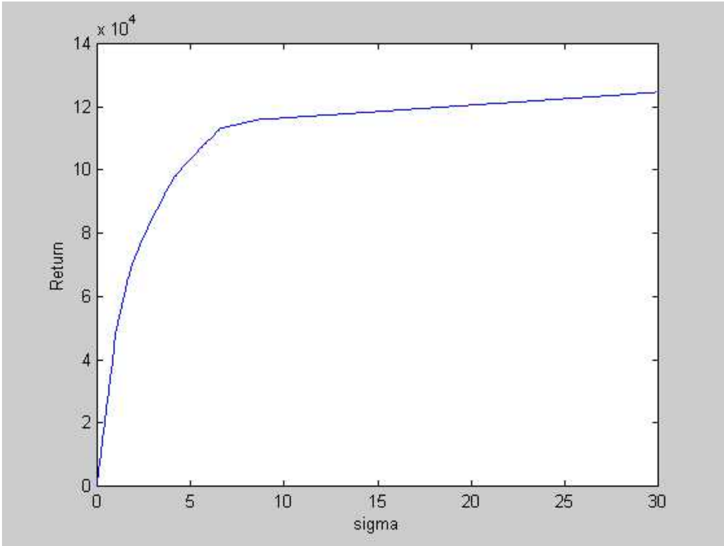

Figure 3. Relation E(σ), σ=0.001 – 30

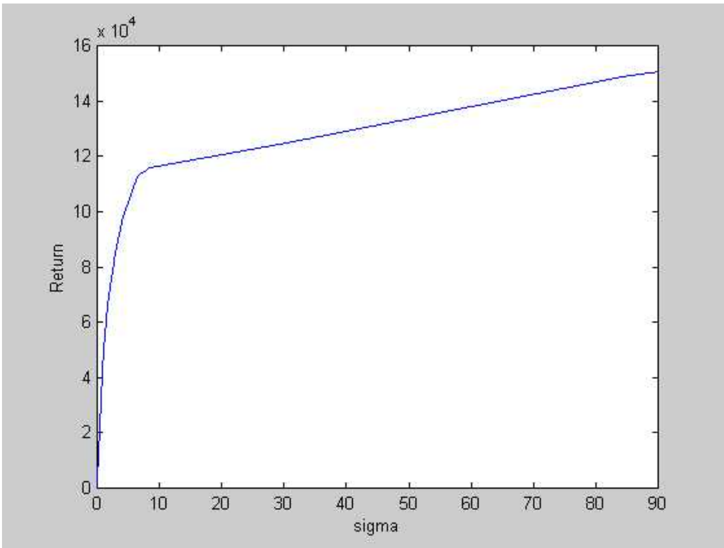

Figure 4. Relation E(σ), σ =0.001 – 90

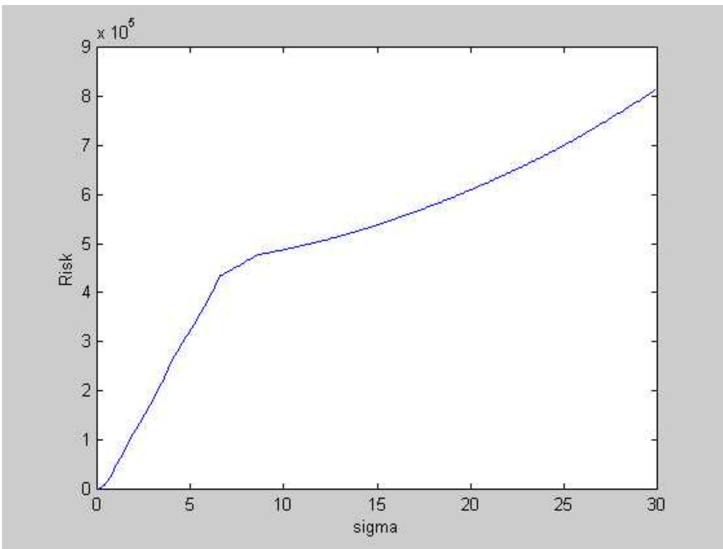

Figure 5. Relation $V(\sigma)$, σ = 0.001 – 30

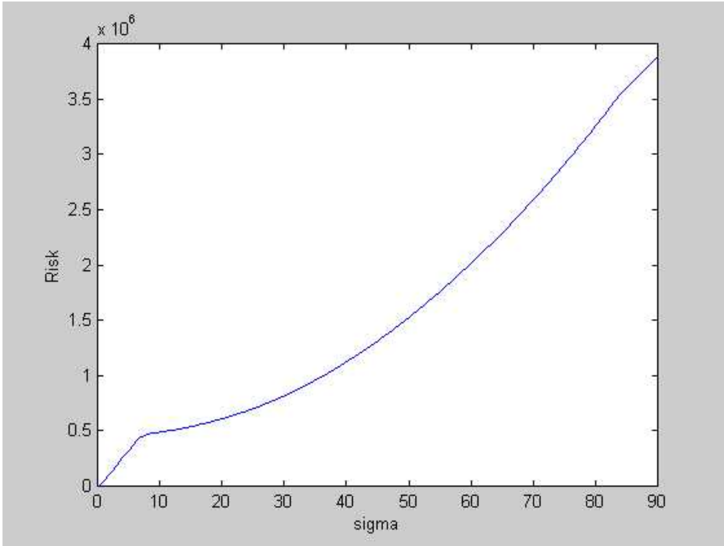

Figure 6. Relation $V(\sigma)$, σ =0.001 – 90

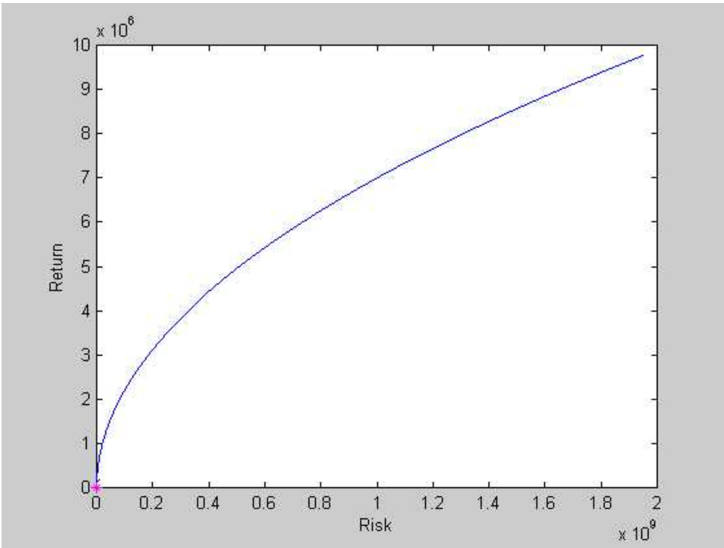

Figure 7. Relation Return-Risk σ=0.001 – 30

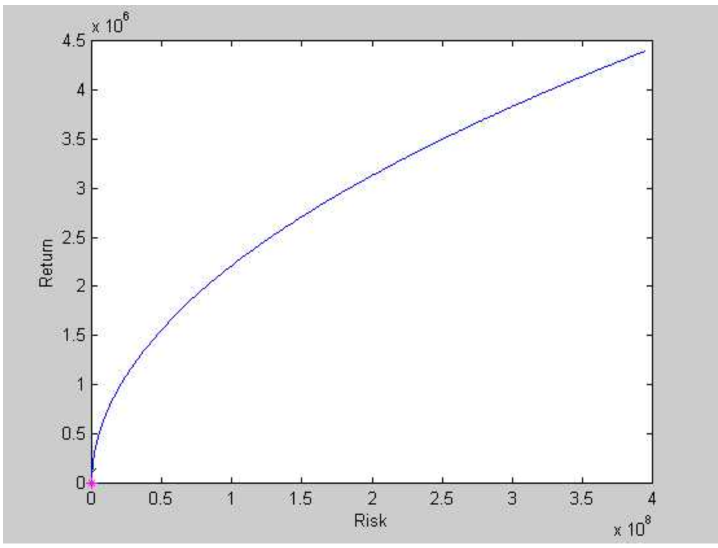

Figure 8. Relation Return-Risk σ=0.001 – 90

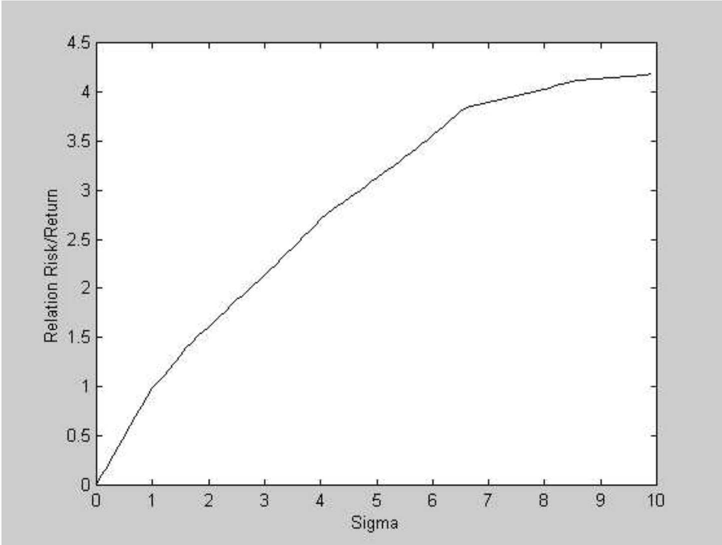

Figure 9. Relation Risk- Return towards σ, σ = 0.001 – 10

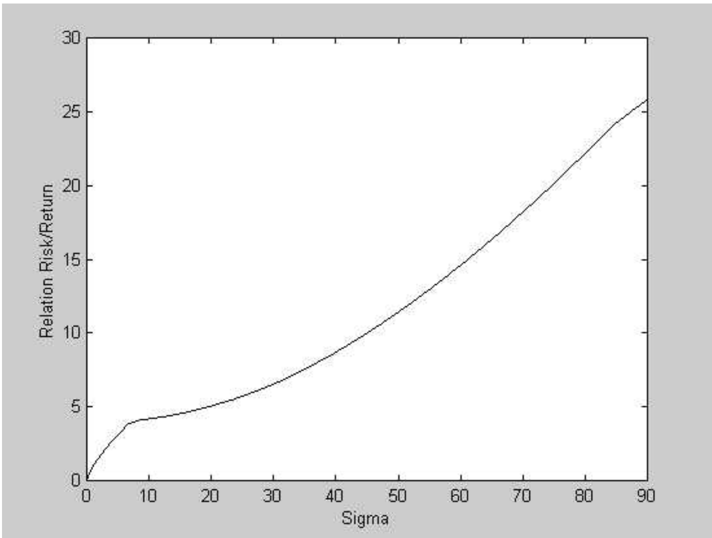

Figure 10. Relation Risk- Return towards σ, σ = 0.001 – 90

$$(15)\quad \sigma^{opt} = \arg\left\{\min_{\sigma}\left\{\frac{V_p(\sigma)}{E_p(\sigma)} = \frac{x^T(\sigma)Qx(\sigma)}{E^T x(\sigma)}\right\}\right\}.$$

Thus, the argument σ is calculated as a solution of a well-defined and consistent optimization problem. In comparison with the classical portfolio theory the value of σ is not assessed by subjective considerations of the financial analyzer, which is an advantage of the bi-level portfolio problem.

Following *figures 9* and *10*, it can be seen that σ_{opt} is situated closed to the origin of the graphics and it has a very small value. Hence, the graphical solution of the Sharp problem (14) cannot be found. Thus, the analytical relation (13), derived in the paper, provides advantages for the optimal policy of investment and quantitative evaluation of the portfolio arguments x_i – investment shares per security i and the coefficient of investor's preferences σ. Hence, σ is found as an argument of the portfolio problem instead of a coefficient, which value is chosen subjectively by the investor.

Conclusions

A bi-level optimization model of the portfolio problem is presented in this paper. The classical solution of the portfolio problem is by one level optimization. The portfolio theory insists that the parameter for risk preference σ be given in advance before solving the portfolio problem. The parameter σ is chosen by the financial analyzer. The estimation of σ is a source of subjective influence for the problem definition and optimal solution. Currently, the portfolio problem is solved for a set of values of σ as a means to estimate the influence of σ for the problem solutions. In this research the process of decision making is presented as a two level optimization system. The upper level defines the optimal value of the parameter of risk preferences of the investor σ by minimizing Sharp ratio (portfolio risk versus portfolio return). The lower optimization level uses *s* and solves the portfolio optimization problem. The bi-level formalism in a unique way defines the most appropriate value of σ by optimizing the Sharp ratio. In that manner, the bi-level formalism achieves two benefits: suppresses the subjective assessment of the investor's risk preferences and calculates and applies the optimal value of σ by minimizing the Sharp ratio. These two outcomes considerably improve the bi-level definition of the portfolio problem in comparison with the classical one level optimization problem.

Additionally, this work develops and applies a special method for solving the optimization problem, titled noniterative coordination. It allows the explicit definition in an analytical manner of the upper level optimization problem for solving s and for eriving explicit analytical relations between the portfolio problem solutions and σ, $x(\sigma)$. These relations speed up the optimal problem solution and the definition of the efficient frontier of portfolios. Thus, the decision making process can be performed in real time which can respond to the fast and dynamic changes of the security market while reducing the risk of investment.

Acknowledgement

This work is partly supported by project INPORT DVU01/0031 funded by "Science Research" of Ministry of Youth, Education and Science.

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Manuscript received on 19.04.2012

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