A Method for Balanced Truncation of Singular Systems

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Key Words: *Singular or descriptor variable system; model order reduction; approximation by residualization; approximation by balanced truncation.*

Abstract. *This paper considers the problem of balanced model reduction for singular systems. The proposed method uses the second canonical form description of the singular or descriptor variable system represented as a combination of dynamic and static equations. The static equations are considered as constraints over the system variables. The method is performed in two steps. First, singular perturbation approximation is used in order to solve for the fast variables in the singular system. These variables are substituted into the dynamical equations representing a regular state space model. Second, balanced truncation on the state space model is performed to reduce the system order. Three special cases are considered depending on the singularity of one of the system block matrices. Different experiments are performed for all three cases showing good approximation properties of the method.*

1. Introduction

Mathematical models are widely used to simulate the dynamical behavior of many physical processes and systems. Sometimes these processes are characterized by very large dimensionality. For example, weather forecasts and very large scale integration circuit simulations may reach hundreds or even thousands coupled differential equations. The need for improved accuracy often leads to models of higher complexity. The simulation of such complicated models is mostly unfeasible task. This is the reason to look for a model simplification, where the computational complexity is mainly reduced. The process of model simplification by decreasing the dimensionality of the primary model is called model reduction. Model reduction is related to deriving low order models from high order ones according to certain criteria. The model reduction problem for linear systems has been well established area and intensively studied recently. In [2] two main approaches for model reduction: state truncation and residualization are presented. The state truncation method is based on the elimination of part of the state vector and using the remaining vector in place of the original one. Given a system description in state space form, we can partition the state vector $x(t)$ into two components: $x_1(t)$ and $x_2(t)$. Along with the state vector partitioning we can also split the system matrices accordingly. The reduced order model can be obtained by eliminating the truncated vector component $x_2(t)$. The resulting truncated system will contain only this part of the system matrices, which corresponds to the state vector component $x_1(t)$. The major advantage of this method is that it preserves stability of the reduced system and its transfer function at infinity is equal to the transfer function of the original system at infinity. The residualization method is based on the singular perturbation approximation procedure. In the singular perturbation approximation the state vector is divided into fast part $x_2(t)$ and slow part $x_1(t)$. The reduced order model is obtained by residualizing $x_2(t)$, accepting that $\dot{x}_2(t)$ is practically zero. Under certain conditions the state vector component $x_2(t)$ is solved with respect to the state vector component $x_1(t)$ and is substituted in the state equation for $\dot{x}_1(t)$. Therefore, the dynamical system model is reduced with the size of the dimension of $x_2(t)$. The major advantage of this method is the preservation of the steady state gain of the original system model. The reduction by truncation provides a reduced order system which approximates the original system well at high frequencies and the residualization method provides a good approximation at low frequencies.

A large class of physical as well as social phenomena can be modeled by using both differential and algebraic equations. Differential equations represent dynamical relations which exist in the system and the algebraic equations are used to describe the constraints and direct connections. Such kind of system models which combine dynamics and statics in its description are called singular or descriptor variable systems. If one deals with large scale singular systems, which are normally expected in practice, then the problem of model reduction becomes important [9]. In [13] two approaches are offered for balanced model reduction of singular systems, which are based on transforming the system into two canonical forms. The first approach uses the singular perturbation approximation technique and the second one uses state truncation. A drawback of the second approach is that it destroys the structure of the canonical form description. A method for singular systems model reduction which preserves the canonical form description is offered in [10]. This method is based on projecting the fast subsystem onto the discrete domain and optimally reducing it to a lower order discrete-time system, then converting it back to a corresponding reduced order fast subsystem. It utilizes the Nehari shuffle algorithm to reduce the fast subsystem and preserves the system structure. The main drawback of the method is that the reduced order fast subsystem is not obtained in balanced form and it also gives a large approximation error in the high – frequency range. Another method for mixed optimal approximation of the fast descriptor subsystem is presented in [17]. The method is based on an optimization procedure over the approximation error, measured as a weighted combination of the Hilbert – Schmidt and the H_2 norms. A method based on the balanced truncation technique and closely related to controllability and observability gramians and Hankel singular values for descriptor systems is presented in [15,7]. The gramians are obtained by solving generalized Lyapunov equations with special right hand sides [3]. The state space is decomposed into complementary deflating subspaces corresponding to the finite and infinite eigenvalues of the pencil $\lambda E - A$. The balanced transformations are performed on the singular system projection on each of the deflating subspaces and in this way defining the Weierstrass – like canonical form of the singular system. The balanced truncation however, is performed only on the proper part, while the improper part is leaved unchanged. Similar approach is proposed in [1], where an extension of the SVD approach for model reduction of the fast descriptor system is implemented with a regular discrete – time Schur algorithm. The projected decomposition structure of the singular system is preserved and stability for both subsystems is also guaranteed. The proposed method shows good agreement between the original and reduced order models for the medium frequency range, but it shows larger deviations for low and high frequencies.

Most of the proposed approaches for balanced model reduction of singular systems are based on the Weierstrass canonical form decomposition of the descriptor system, i.e. the decomposition on fast and slow subsystems. However, the singular system is often presented as a combination of dynamical and statical relations, which is associated with the practical considerations that certain constraints on the state variables influence the system dynamics. Using canonical form decomposition into dynamic and static subsystems, the singular perturbation approximation method is the natural approach for model order reduction. In [11] it is shown that the direct truncation reduction and the slow singular perturbation approximation of a stable internally balanced system are two fully compatible model reduction methods which give the same upper bounds on the approximation error. Moreover, the reduction by singular perturbation approximation is the natural choice for singular systems model reduction.

This paper considers the problem of balanced model reduction for stable singular systems. The proposed method is an extension of the first approach for model reduction of singular systems in [13], which considers the special case when the block system matrix for the fast state variables is not invertible. Opposite to the approach in [11], where the singular perturbation approximation is performed over the already balanced system, the proposed approach implements singular perturbation approximation first and after that balancing transformation and truncation. Singular perturbation approximation is especially fitted for model reduction of singular systems because state vector derivatives of the fast state variables is actually zero, which is due to the existence of static relations in the system description. The paper considers also the special case when the singular system is transformed into a Weierstrass canonical form and the fast state variable is missing from the algebraic equation description.

2. Linear Dynamical Systems in Descriptor Form

The linear, time – invariant, continuous – time singular (descriptor variable) system is described by the following equations:

(1.1) $E\dot{x}(t) = Ax(t) + Bu(t)$;

(1.2) $y(t) = Cx(t)$.

Important feature of these models is the fact that E is a singular matrix. One special case is the singularly perturbed system:

(2.1) $\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)$;

(2.2) $\varepsilon\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t)$.

When $\varepsilon = 0$ one may solve the resulting algebraic constraints for $x_2(t)$ and eliminate it in order to obtain equations for the slow subsystem. It is well known however, that the properties for small ε are not determined solely in terms of the slow subsystem but that the fast subsystem must also be taken into account. Thus conversion of a singular system into a state variable system can be accompanied by a loss of information.

Consider the singular system described by (1). In [5] is shown that for any two matrices E and A there always exist two nonsingular matrices Q and P, such that the singular system will be regular if and only if

(3) $QEP = diag\,(I, N)$; $QAP = diag\,(A_1, I)$.

Because of the fact that this condition is difficult to verify an easier test for regularity is the following definition [5].

Definition 1. For any two matrices $E, A \in R^{n\times n}$, the pencil (E, A) is called regular if there exists a constant scalar $\alpha \in C$, such that

(4) $\mathbf{det}\,(\alpha E + A) \neq 0$ or $\mathbf{det}\,(sE - A) \neq 0, \forall \alpha, s$.

Definition 2. A singular system (E, A, B, C) is called restricted system equivalent to a system $(\bar{E}, \bar{A}, \bar{B}, \bar{C})$ if there exist two nonsingular matrices Q and P such that

(5) $x = P\bar{x}$, $QEP = \bar{E}$, $QAP = \bar{A}$, $QB = \bar{B}$, $CP = \bar{C}$.

The restricted equivalence preserves the structure of the impulsive modes corresponding to the free response of the system [5]. It is often much more convenient to work with special forms of restricted system equivalence, called canonical forms of singular systems. Basic requirement for their existence is the regularity condition.

First Canonical Form. Assume that the singular system (1) is regular. Then there exist two nonsingular matrices Q and P, such that the following decomposition holds:

(6.1) $\dot{x}_1(t) = A_1x_1(t) + B_1u(t)$;

(6.2) $y_1(t) = C_1x_1(t)$;

(6.3) $N\dot{x}_2(t) = x_2(t) + B_2u(t)$;

(6.4) $y_2(t) = C_2x_2(t)$;

(6.5) $y(t) = C_1x_1(t) + C_2x_2(t)$;

where

$x_2(t) \in R^{n_2}, x_1(t) \in R^{n_1}, QEP = diag\,(I, N), QAP = diag\,(A_1, N)$,

$QB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, $CP = (C_1 \quad C_2)$.

Matrix N is a nilpotent matrix with order of nilpotency h. This form is also known as Weierstrass form [15]. The subsystems obtained from the decomposition of the original one are called slow and fast subsystems.

Second Canonical Form. Consider the singular system (1) which is regular. Then there exist nonsingular matrices Q and P such that

(7) $QEP = diag\ (I_q,\ 0)$, $P^{-1}x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x_1 \in R^q$, $x_2 \in R^{n-q}$.

The original system is restricted system equivalent to

(8.1) $\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)$;

(8.2) $0 = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t)$;

(8.3) $y(t) = C_1x_1(t) + C_2x_2(t)$

where

(9) $QAP = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, QB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, $CP = (C_1 \quad C_2)$.

This canonical form clearly reflects the physical meaning of the singular systems, i.e. the mixed dynamics and statics. The singular value decomposition is usually applied to transform the original system into the second canonical form and therefore, the transformation matrices are specified from the orthogonal matrices of the SVD algorithm.

The slow subsystem from the first canonical form represents an ordinary differential equation, which has a unique solution with initial condition $x_1(0)$ given as follows:

(10) $x_1(t) = e^{A_1t}x_1(0) + \int_0^t e^{A_1(t-\tau)}B_1u(\tau)d\tau$

where $x_1(t)$ is completely determined by $x_1(0)$ and $u(\tau)$, $0 \le \tau \le t$. Let us assume that $u(t)$ is h times continuously differentiable, where h is the index of nilpotency of the matrix N. Then the response of the fast subsystem is given by:

(11) $x_2(t) = -\sum_{i=0}^{h-1} N^iB_2u^{(i)}(t)$.

There are values of $x_2(0)$ which yield impulsive solutions for $x_2(t)$. Since discontinuous behavior is not desirable, the set of $x(0)$ which do not result in such behavior at $t = 0$ is called the set of admissible (consistent) initial conditions. In the case of fast change of the system states at $t = 0$ however, a new term is presented in the response which contains delta impulses and derivatives of delta impulses up to the order of the nullity of N minus one. This solution, called distributive solution can be represented by the equation [16]:

(12) $x_2(t) = -\sum_{i=1}^{h-1} \delta^{(i-1)}(t)N^ix_2(0) - \sum_{i=0}^{h-1} N^iB_2u^{(i)}(t)$.

This type of solution corresponds to many practical problems where sudden change of the state can happen. The transfer function of the singular system (1) is defined as:

(13) $G(s) = C(sE - A)^{-1} B = C_1(sI - A_1)^{-1} + C_2(sN - I)^{-1} B_2$.

The transfer function is system invariant, i.e. it is preserved under a system equivalence transformation.

Definition 3. The pencil $\lambda E - A$ is called *c-stable* if it is regular and all finite eigenvalues of $\lambda E - A$ lie in the open left half – plane. Thus, the singular system is stable if and only if its slow subsystem is stable.

Definition 4. The singular system is called *R-controllable* if it is controllable in the reachable set or more precisely for any prescribed $t_1 > 0$, $x_1(0) \in R$ and $\omega \in R$, there always exists an admissible control input $u(t)$, such that $x(t) = \omega$. The following are equivalent:

The singular system is R – controllable

The slow subsystem is controllable

rank $[sE - A \quad B] = n$ for every finite $s \in C$

rank $(B_1 \quad A_1B_1 \ldots A_1^{n_1-1}B_1) = n_1$

Definition 5. The singular system is called *impulse controllable* if for any initial condition $x(0)$, $\tau \in R$ and $\omega \in R^{n_2}$, there exists an admissible control input $u(t)$, such that

$x_\tau(t) = \begin{pmatrix} 0 \\ I_{2\tau}(\omega,t) \end{pmatrix}$, $I_{2\tau}(\omega,t) = \sum_{i=1}^{h-1} \delta^{(i-1)}(t-\tau)N^i\omega$.

The following statements are equivalent:

The singular system is *impulse controllable*

The fast subsystem is *impulse controllable*

Ker $(N) = Range(B_2 \quad Nb_2 \ldots N^{h-1}B_2) = R^{n_2}$

Range $(N) = Range\ (B_2) = Ker\ (N) = R^{n_2}$.

The slow subsystem is always *impulse controllable* and the fast subsystem is always *R – controllable* [4].

Definition 6. The singular system is *R – observable* if it is observable in the reachable set or any state in the reachable set may be uniquely determined by $y(t)$ and $u(\tau)$, $0 \le \tau \le t$. The following statements are equivalent:

The singular system is *R – observable*

The slow subsystem is observable

The matrix $T = \begin{pmatrix} -A & E & 0 & \cdots & 0 \\ 0 & -A & E & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C & 0 & 0 & \cdots & 0 \\ 0 & C & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$ is of full column rank.

Definition 7. The singular system is *impulse observable* if the impulse behavior of the state is uniquely determined from the information of the impulse behavior in the output and the input. The following statements are equivalent:

The system is *impulse observable*

The fast subsystem is *impulse observable*

$Ker(N) \cap Ker(C_2) \cap Range(N) = \{0\}$

$$rank\begin{pmatrix} E & A \\ 0 & E \\ 0 & C \end{pmatrix} = n + rank(E).$$

By analogy to controllability of singular systems, the fast subsystem is always *R – observable* and the slow subsystem is always *impulse observable* [4].

Important concepts for the singular systems are the controllability and observability gramians [15,7]. Assume that the singular system (1) is *c-stable* and is transformed in its Weierstrass form by using similarity transformations as follows:

(14) $E = Q^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}P^{-1}$ and $A = Q^{-1}\begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}P^{-1}$.

Then the integrals:

(15) $W_{pc} = \int_0^\infty F(t)BB^T F^T(t)dt$ and $W_{po} = \int_0^\infty F^T(t)C^T CF(t)dt$

exist, where $F(t) = P\begin{bmatrix} e^{A_1 t} & 0 \\ 0 & 0 \end{bmatrix}Q$. The matrix W_{pc} is called the proper controllability gramian and the matrix W_{po} is called the proper observability gramian [15,7]. The improper controllability and observability gramians are defined as:

(16) $W_{ic} = \sum_{k=-h}^{-1} F_k BB^T F_k^T$ and $W_{io} = \sum_{k=-h}^{-1} F_k^T C^T CF_k$

where $F_k = P\begin{bmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{bmatrix}Q$. If $E = I$, the proper gramians are the usual gramians for the standard state space system.

The proper controllability and observability gramians are the unique symmetric positive semidefinite solutions of the projected generalized continuous – time Lyapunov equations [15,3]:

(17) $EW_{pc}A^T + AW_{pc}E^T = -P_l BB^T P_l^T$, $W_{pc} = P_r W_{pc}$

(18) $E^T W_{po}A + A^T W_{po}E = -P_r^T C^T CP_r$, $W_{po} = W_{po}P_l$

where $P_r = P\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$ and $P_l = Q^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}Q$ are the spectral projections onto the right and left deflating subspaces of $\lambda E - A$ corresponding to the finite eigenvalues. The improper controllability and observability gramians are the unique symmetric positive semidefinite solutions of the projected generalized discrete – time Lyapunov equations:

(19) $AW_{ic}A^T - EW_{ic}E^T = (I - P_l)BB^T(I - P_l)^T$; $P_r W_{ic} = 0$

(20) $A^T W_{io}A - E^T W_{io}E = (I - P_r)^T C^T C(I - P_r)$, $W_{io}P_l = 0$.

Definition 8. Consider the singular system (1), where the pencil $\lambda E - A$ is *c-stable*.

The system (1) is R – controllable and R – observable if and only if

(21.1) *rank* (W_{pc}) = *rank* $(W_{po}) = n_1$.

The system (1) is impulse – controllable and impulse – observable if and only if

(21.2) *rank* (W_{ic}) = *rank* $(W_{io}) = n_2$.

The system (1) is c – controllable and c – observable if and only if (21.1) and (21.2) both hold.

Definition 9. The singular system (1) is minimal if it is c-controllable and c-observable.

Definition 10. A realization (E, A, B, C) of the transfer function $G(s)$ is called balanced [15], if the following relations hold:

(22) $W_{pc} = W_{po} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$ and $W_{ic} = W_{io} = \begin{bmatrix} 0 & 0 \\ 0 & \Theta \end{bmatrix}$

with $\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_{n_1})$ and $\Theta = diag(\theta_1, \theta_2, ..., \theta_{n_2})$. For a minimal realization (E, A, B, C) with the *c-stable* pencil $\lambda E - A$, there exists a system equivalence transformation (Q_b, P_b) such that the realization $(Q_b EP_b, Q_b AP_b, Q_b B, CP_b)$ is balanced. Computing the reduced – order singular system can be interpreted as performing a system equivalence transformation $(\widetilde{Q}, \widetilde{P})$ such that

(23) $$\left[\begin{array}{c|c} \widetilde{Q}(sE - A)\widetilde{P} & \widetilde{Q}B \\ \hline C\widetilde{P} & \end{array}\right] = \left[\begin{array}{cc|c} sE_f - A_f & & B_f \\ & sE_\infty - A_\infty & B_\infty \\ \hline C_f & C_\infty & 0 \end{array}\right]$$

where the pencil $\lambda E - A$ has the finite eigenvalues only, all the eigenvalues of the pencil $\lambda E_\infty - A_\infty$ are infinite, and then reducing the order of the subsystems $\lfloor E_f, A_f, B_f, C_f \rfloor$ and $[A_\infty, E_\infty, B_\infty, C_\infty]$ with nonsingular E_f and A_∞ using classical balanced truncation methods for continuous – time and discrete – time state space systems, respectively [15,1]. The approximation error is computed from the expression:

(24) $$\left\| G(s) - \widetilde{G}(s) \right\|_\infty \le 2(\sigma_{k+1} + \ldots + \sigma_{n_1})$$

where $G(s)$ and $\widetilde{G}(s)$ are the original and reduced k^{th} order strictly proper transfer functions corresponding to $\lfloor E_f, A_f, B_f, C_f \rfloor$, since the polynomial transfer functions corresponding to $[A_\infty, E_\infty, B_\infty, C_\infty]$ are equal, i.e. $P(s) = \widetilde{P}(s)$ [15].

3. Direct Method for Balanced Truncation of Singular Systems

The direct method for model order reduction of singular systems is based on the second canonical form description. This method is an extension of the method proposed in [13], for the cases when the second block system matrix in the static equation is not invertible. This method implements singular perturbation approximation for singular systems where the fast subsystem is described by algebraic equation. Therefore, the method is exact in some sense because the fast variable derivative is actually zero. Consider the singular system (1). Assume that the pencil $\lambda E - A$ is regular and therefore its second canonical form (8) exists with dynamic equations of order n_1 and static equations of order n_2. The transformation matrices P and Q are not unique and one way of performing the decomposition is by applying the algorithm of singular value decomposition of the matrix E, i.e.

(25) $U^T EV = diag(\Sigma,0)$

and using the coordinate transformation $V^T x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ we obtain the matrices:

(26) $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = TU^T AV$, $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = TU^T B$,

$\begin{pmatrix} C_1 & C_2 \end{pmatrix} = CV$, $T = \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & I \end{pmatrix}$.

The algebraic equation plays a role of constraints over the state variables. In this case we isolate the algebraic equations in order to decompose the singular system. The decomposition itself reduces the order of the system.

Case I. Assume that the matrix A_{22} is nonsingular. The nonsingularity means that both of the subsystems, the dynamic as well as the static one, can be decoupled. We solve the static equation with respect to the state variable $x_2(t)$ and obtain a system in which the dynamic variables are separated from the static ones as follows:

(27.1) $\dot{x}_1(t) = \widetilde{A}_1 x_1(t) + \widetilde{B}_1 u(t)$;

(27.2) $y(t) = \widetilde{C}_1 x_1(t) + \widetilde{D}_1 u(t)$;

(27.3) $x_2(t) = \widetilde{A}_2 x_1(t) + \widetilde{B}_2 u(t)$,

where

(28.1) $\widetilde{A}_1 = A_{11} - A_{12} A_{22}^{-1} A_{21}$;

(28.2) $\widetilde{B}_1 = B_1 - A_{12} A_{22}^{-1} B_2$;

(28.3) $\widetilde{C}_1 = C_1 - C_2 A_{22}^{-1} A_{21}$;

(28.4) $\widetilde{D}_1 = -C_2 A_{22}^{-1} B_2$;

(28.5) $\widetilde{A}_2 = -A_{22}^{-1} A_{21}$;

(28.6) $\widetilde{B}_2 = -A_{22}^{-1} B_2$.

If the singular system (1) is regular, stable, controllable and observable and matrix A_{22} from (8) is invertible, then the system (27) is stable, controllable and observable [13]. The system model (27) is a standard state space dynamical system model and we can apply balancing transformations [8,14] in order to convert the system state variables into a balanced form. Then the method of balanced truncation can be applied to reduce the order of the system model.

Case II. Assume that the matrix A_{22} is singular but not the zero matrix. As pointed out in [16], the singularity of A_{22} leads to the appearance of impulsive modes in the state equation solution of (8). However, if we make the assumption that the system (1) is c-controllable and c-observable, then the fast subsystem which causes the appearance of impulsive motions in the state response, corresponding to the polynomial part of the transfer function $G(s)$, can not be reduced as mentioned in [15]. Only the state realization of the strictly proper part of $G(s)$ is subjected to balanced truncation and model order reduction. In this case we can use the Moore – Penrose generalized inverse for A_{22} to solve the static equation with respect to the state variable $x_2(t)$ [12]. The Moore – Penrose generalized inverse of a given matrix A, denoted by $A^\dagger$ possesses some interesting properties [12,6]:

(29) $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^T = AA^\dagger$, $(A^\dagger A)^T = A^\dagger A$.

If the matrix A is invertible, then $A^\dagger = A^{-1}$ and finally if we look for the solution of the linear system algebraic equations $Ax = b$ where A is not invertible, the solution $x = A^\dagger b$ is the one of minimal 2 – norm [6]. Therefore, the Moore – Penrose generalized inverse minimizes the operator induced two norm of $(AA^\dagger - I)$ or $(A^\dagger A - I)$. The Moore – Penrose generalized inverse of a given matrix A is computed as follows. Obtain the singular value decomposition of a matrix A as $A = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}V^T$, where the block-matrix Σ contains the nonzero singular values of A, and then calculate $A^\dagger = V\begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^T$.

After solving the algebraic equation with respect to the variable $x_2(t)$, the obtained dynamical system is (27) with the corresponding matrices:

(30.1) $\widetilde{A}_1 = A_{11} - A_{12} A_{22}^\dagger A_{21}$;

(30.2) $\widetilde{B}_1 = B_1 - A_{12} A_{22}^\dagger B_2$;

(30.3) $\widetilde{C}_1 = C_1 - C_2 A_{22}^\dagger A_{21}$;

(30.4) $\widetilde{D}_1 = -C_2 A_{22}^\dagger B_2$;

(30.5) $\widetilde{A}_2 = -A_{22}^\dagger A_{21}$;

(30.6) $\widetilde{B}_2 = -A_{22}^\dagger B_2$.

Then, the method of balanced truncation can be applied to reduce the order of the system model (27).

The following algorithm presents the procedure for balanced model reduction of a singular system (1) by utilizing its second canonical form description.

Algorithm for Direct Balanced Truncation of Singular Systems

Step1. Use SVD (25) and apply similarity transformations TU^T and V from (26) to obtain the system in its second canonical form (8) (if it is not initially presented in this form).

Step 2. Use SVD for the matrix A_{22}, i.e. $A_{22} = U\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}V^T$ and obtain its Moore – Penrose generalized inverse $A_{22}^{\dagger} = V\begin{bmatrix}\Sigma^{-1} & 0\\ 0 & 0\end{bmatrix}U^T$ (if the matrix A_{22} is invertible, then $A^{\dagger} = A^{-1}$).

Step 3. Apply the method of residualization and obtain the standard state space dynamical system (27) with system matrices computed as in (30).

Step 4. Apply the square root algorithm from [2] for balancing the system (27)

- $W_o = LL^*$ (Cholesky decomposition)
- $W_c = UU^*$ (Cholesky decomposition)
- $U^*L = W\Sigma V^*$ (SVD decomposition)
- $P = \Sigma^{-1/2}V^*L^*$ (similarity transformation)
- $P^{-1} = UW\Sigma^{-1/2}$ (similarity transformation)

where W_c and W_o are the controllability and observability gramians for the system (27), correspondingly.

Step 5. Apply the method of truncation to reduce the order of the system model (27).

Case III. Matrix A_{22} is a zero matrix. This is usually the case when the singular system is transformed in its first canonical (Weierstrass) form (6). Consider only the fast subsystem (6.3)-(6.4). We take the following example from [18]:

$$N = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix},\ \dot{x}_2(t) = \begin{bmatrix}\dot{x}_{21}(t)\\ \dot{x}_{22}(t)\\ \dot{x}_{23}(t)\end{bmatrix},\ B_2 = \begin{bmatrix}b_{21}\\ b_{22}\\ b_{23}\end{bmatrix}.$$

The obtained dynamical equations are as follows:

$\dot{x}_{22}(t) = x_{21}(t) + b_{21}u(t);$

$\dot{x}_{23}(t) = x_{22}(t) + b_{22}u(t);$

$0 = x_{23}(t) + b_{23}u(t).$

Solving the equations backwards, we obtain:

$x_{23}(t) = -b_{23}u(t);$

$x_{22}(t) = -b_{23}\dot{u}(t) - b_{22}u(t);$

$x_{21}(t) = -b_{23}\ddot{u}(t) - b_{22}\dot{u}(t) - b_{21}u(t).$

As can be seen, the solution does not depend on the initial conditions and depends solely on the input signal and its derivatives. In order to reduce the order of the system model it is necessary to reduce the orders of the input signal derivatives. However, the input signal is an exogenous signal and is independent on system description. This example shows that reducing the model order of the fast subsystem is often an unfeasible task. Let us consider now the general case where the fast subsystem $N\dot{x}(t) = x(t) + Bu(t)$ is of order n_2. We can perform singular value decomposition of matrix N as follows: $N = U\Sigma V^T$, where matrix U is the identity matrix, matrix Σ and matrix V have the following forms:

$$(31)\quad \Sigma = \begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix},\ V = \begin{bmatrix}0 & 0 & \cdots & 0 & \pm 1\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix},$$

where $V(1,n_2) = 1$, if n_2 is an odd number and $V(1,n_2) = -1$, if is an even number. By using state transformation $\tilde{x} = V^T x$ and partitioning the state vector as $\tilde{x}(t) = \begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix}$, where $\tilde{x}_1(t) \in R^{n_2-1}$ and $\tilde{x}_2(t) \in R$, we obtain the system description $\Sigma\,\dot{\tilde{x}}(t) = Vx(t) + Bu(t)$, which in scalar form is presented as:

$$(32.1)\quad \dot{\tilde{x}}_{11}(t) = \pm\tilde{x}_2(t) + b_1 u(t);$$

$$(32.2)\quad \dot{\tilde{x}}_{12}(t) = \tilde{x}_{11}(t) + b_2 u(t);$$

$$(32.3)\quad \dot{\tilde{x}}_{13}(t) = \tilde{x}_{12}(t) + b_3 u(t);$$

$$\vdots$$

$$(32.4)\quad 0 = \tilde{x}_{1n_2-1}(t) + b_{n_2}u(t).$$

Obviously matrix A_{22} is zero since the state variable $\tilde{x}_2(t)$ is not presented in the last equation. Moreover, $\tilde{x}_2(t)$ appears only in the first equation and its derivative is not presented in the equations left hand side. We consider two cases: *i)* $u(t)$ is a scalar function and *ii)* $u(t)$ is a vector. In the first case, if $b_{n_2} \neq 0$, we can solve the last equation with respect to u and obtain:

$$(33)\quad u(t) = -\frac{1}{b_{n_2}}\tilde{x}_{1n_2-1}(t).$$

Substituting for $u(t)$, the equations (32) then become:

$$(34.1)\quad \dot{\tilde{x}}_{11}(t) = -\frac{b_1}{b_{n_2}}\tilde{x}_{1n_2-1} \pm \tilde{x}_2(t);$$

$$(34.2)\quad \dot{\tilde{x}}_{12}(t) = \tilde{x}_{11}(t) - \frac{b_2}{b_{n_2}}\tilde{x}_{1n_2-1}(t);$$

(34.3) $\dot{\tilde{x}}_{13}(t)=\tilde{x}_{12}(t)-\frac{b_3}{b_{n_2}}\tilde{x}_{1n_2-1}(t)$.

$$\vdots$$

Accepting $\tilde{x}_2(t)$ as exogenous input signal, we obtain the equations:

(35.1) $\dot{\tilde{x}}_1(t)=\tilde{A}_1\tilde{x}_1(t)\pm e_1\tilde{x}_2(t)$;

(35.2) $y(t)=CV\begin{bmatrix} x_1(t) \\ \tilde{x}_2(t) \end{bmatrix}$

where

$$(36)\qquad \tilde{A}_1=\begin{bmatrix} 0 & 0 & \cdots & 0 & -\frac{b_1}{b_{n_2}} \\ 1 & 0 & \cdots & 0 & -\frac{b_2}{b_{n_2}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\frac{b_{n_2-1}}{b_{n_2}} \end{bmatrix},\quad e_1=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The system (35) as a regular state space dynamical system and we can apply balancing truncation techniques to reduce the system model order.

The second case is when $u(t)$ is a vector function of time. In this case the input signal can not be determined from the last equation (32.4) because b_{n_2} is a vector – row and is not invertible. However, we can use its Moore – Penrose generalized inverse, which is calculated as follows. Calculate the singular value decomposition of b_{n_2} as $b_{n_2}=PSQ^T$, where S is a vector row with all elements zero except the first element s_1. Then $b_{n_2}^\dagger$ is a vector column computed as $b_{n_2}^\dagger=Q\begin{bmatrix} s_1^{-1} & 0 & \cdots & 0\end{bmatrix}^T P^T$. Then

(37) $u(t)=-b_{n_2}^\dagger\tilde{x}_{1n_2-1}(t)$;

(38.1) $\dot{\tilde{x}}_{11}(t)=-b_1 b_{n_2}^\dagger\tilde{x}_{1n_2-1}(t)\pm\tilde{x}_2(t)$;

(38.2) $\dot{\tilde{x}}_{1n_2-1}(t)=\tilde{x}_{1n_2-2}-b_{n_2-1}b_{n_2}^\dagger\tilde{x}_{1n_2-1}(t)$;

$$\vdots$$

(38.3) $\dot{\tilde{x}}_{1n_2-1}(t)=\tilde{x}_{1n_2-2}-b_{n_2-1}b_{n_2}^\dagger\tilde{x}_{1n_2-1}(t)$.

where b_i, $i = 1, 2, ..., n_2-1$ is the i^{th} vector row of matrix B. The equations (38) describe the standard state space system (35), where $\tilde{x}_2(t)$ can be considered as exogenous input signal and the matrix $\tilde{A}_1$ and vector – column e_1 are:

$$(39)\qquad \tilde{A}_1=\begin{bmatrix} 0 & 0 & \cdots & 0 & -b_1 b_{n_2}^\dagger \\ 1 & 0 & \cdots & 0 & -b_2 b_{n_2}^\dagger \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -b_{n_2-1}b_{n_2}^\dagger \end{bmatrix},\quad e_1=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

4. Experimental Results

Example I. Consider a singular system (1) with matrices:

$$E=\begin{bmatrix} 0 & 0 & 0.1 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad A=\begin{bmatrix} 1 & 50 & -1 & -50 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix},$$

$$B=\begin{bmatrix} 2 \\ 20 \\ 0 \\ 20 \end{bmatrix},\quad C=\begin{bmatrix} -3 \\ 40 \\ 4 \\ -41 \end{bmatrix}^T.$$

Applying step 1 of the algorithm, we transform the system into its second canonical form (8), where the system matrices are computed as follows:

$$A_{11}=\begin{bmatrix} -0.1078 & -0.0883 & -0.1394 \\ -1.0883 & -0.8922 & -1.4073 \\ 348.6628 & -30.9716 & -5.0 \end{bmatrix},\quad A_{12}=\begin{bmatrix} -0.0153 \\ 1.2805 \\ -5.0 \end{bmatrix},$$

$$A_{21}=\begin{bmatrix} -50.205 \\ 5.4737 \\ 0.7071 \end{bmatrix}^T,\ A_{22}=0.7071,\ B_1=\begin{bmatrix} 2.7871 \\ 28.1466 \\ 10.0 \end{bmatrix},\ B_2=-1.4142,$$

$$C_1=\begin{bmatrix} 56.9272 \\ -6.3475 \\ 4.0 \end{bmatrix}^T,\quad C_2=3.0.$$

As can be observed, matrix A_{22} is nonsingular (it is a scalar different than zero) and therefore can be inverted, so we have case I. Applying steps 2 and 3 of the algorithm we obtain the standard dynamical system (27) with matrices:

$$\tilde{A}_1=\begin{bmatrix} -1.1952 & 0.0302 & -0.124 \\ 89.8302 & -10.8048 & -2.6879 \\ -6.3399 & 7.7334 & 0 \end{bmatrix},\quad \tilde{B}_1=\begin{bmatrix} 2.7564 \\ 30.7077 \\ 0 \end{bmatrix},$$

$$\tilde{C}_1=\begin{bmatrix} 269.93 \\ -29.5705 \\ 1.0 \end{bmatrix}^T,\ \tilde{D}_1=6.0,\ \tilde{A}_2=\begin{bmatrix} 71.0 & -7.741 & -1.0 \end{bmatrix},$$

$\tilde{B}_2=2.0$.

Therefore, by using residualization the order of the system model is reduced by one. The Hankel singular values of the system (27) are calculated as:

$$S=\{109.91,\quad 64.286,\quad 2.27\}.$$

After applying step 4 of the algorithm the system is transformed into a balanced form. The step responses for the full order, reduced second order and reduced first order system models are shown on *figure 1*.

From *figure 1* it can be observed that the third order and the reduced second order system models have almost the same step responses. The step response of the reduced first order model differs significantly from the others. Similarly on *figure 2*, the impulse responses of the third and reduced second order models are very close. The impulse response of the reduced first order model deviates from the others considerably. Therefore, the original fourth order singular system model can be successfully reduced to e second order model.

Example II. Consider the singular system (1) with system matrices as follows:

$$E = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ -8 & -5 & -2 & 1 & 0 \\ 1 & 0 & 1 & -2 & -1 \\ 0 & 1 & 0 & 4 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 10 \\ 5 \\ 2 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 1 \\ 3 \end{bmatrix}^T.$$

Applying step 1 of the algorithm, we transform the system into its second canonical form (8), where the system matrices are computed as follows:

$$A_{11} = \begin{bmatrix} 0 & 0.125 & 0 \\ 0 & 0 & 0.2 \\ -4.0 & -2.5 & -1.0 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 0.25 & 0.125 \\ 0 & 0.2 \\ 0.5 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.125 \\ 2.0 \\ 2,5 \end{bmatrix}, \; B_2 = \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix}, \; C_1 = [3.0 \;\; 4.0 \;\; 2.0], \;\; C_2 = [1.0 \;\; 3.0].$$

As can be observed matrix A_{22} is singular, it can not be inverted and therefore we have case II. The Moore – Penrose generalized inverse of A_{22} is calculated as follows:

$$A_{22}^{+} = V\begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T = \begin{bmatrix} 0.8944 & -0.4472 \\ 0.4472 & 0.8944 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.4472 & 0.8944 \\ 0.8944 & 0.4472 \end{bmatrix} = \begin{bmatrix} -0.08 & 0.16 \\ -0.04 & 0.08 \end{bmatrix}$$

Applying steps 2 and 3 of the algorithm, we obtain the standard dynamical system (27) with matrices:

$$\widetilde{A}_1 = \begin{bmatrix} 0.025 & 0.075 & 0.025 \\ 0.008 & -0.016 & 0.208 \\ -3.96 & -2.58 & -0.96 \end{bmatrix}, \quad \widetilde{B}_1 = \begin{bmatrix} 0.125 \\ 2.0 \\ 2.5 \end{bmatrix}, \quad \widetilde{C}_1 = \begin{bmatrix} 3.2 \\ 3.6 \\ 2.2 \end{bmatrix}^T,$$

$$\widetilde{D}_1 = 0, \; \widetilde{A}_2 = \begin{bmatrix} 0.08 & -0.16 & 0.08 \\ 0.04 & -0.08 & 0.04 \end{bmatrix}, \; \widetilde{B}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, by using residualization the order of the system model is reduced by two. The Hankel singular values of the system (27) are calculated as:

$$S = \{12.556, \;\; 8.251, \;\; 1.0175\}.$$

After applying step 4 of the algorithm the system is transformed into a balanced form. The step responses for the full order, reduced second order and reduced first order system models are shown on *figure 3*.

The step responses of the full order and reduced second order system models are closely related. The step response of the reduced first order model deviates considerably from the other two responses. Comparisons between the full order, the reduced second order and the reduced first order models for an impulse input signal can be seen on *figure 4*.

Example III. Consider the fast subsystem from the first canonical form description (6.3) – (6.4) with the following system matrices:

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}^T.$$

Here we have case III, where after applying singular value decomposition of matrix E and the corresponding change of coordinates, the system is transformed into the form (35) with matrices:

$$\widetilde{A}_1 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \widetilde{C}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T, \quad \widetilde{C}_2 = 0.$$

Therefore, by using residualization the order of the system model is reduced by one. The Hankel singular values of the obtained regular dynamical system are calculated as:

$$S = \{2.4 \;\;\; 0.914 \;\;\; 0.0114\}.$$

After applying step 4 of the algorithm the system is transformed into a balanced form. The step responses for the full order, reduced second order and reduced first order system models are shown on *figure 5*.

The step response of the reduced first order system deviates substantially from the step responses of the full order and reduced second order systems. Similar behavior is observed on *figure 6*, where impulse responses of the full order, reduced second and first order systems are shown.

5. Conclusion

This paper considers the problem of balanced truncation and model reduction for singular systems. The proposed method uses the second canonical form

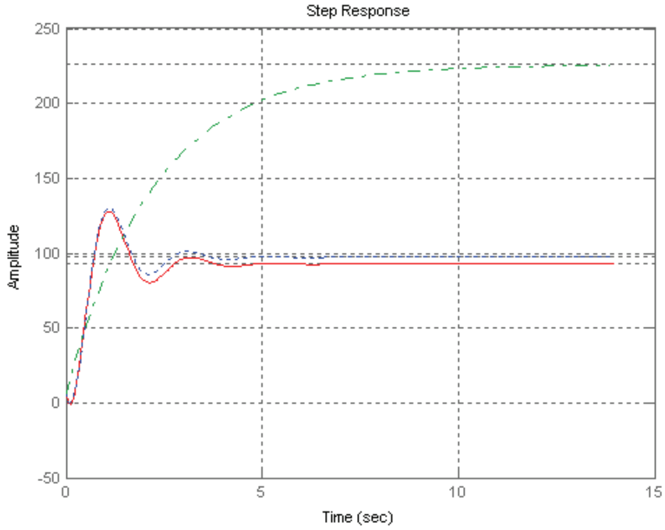

Figure 1. Step response of the full order model -----, reduced 2nd order ..... and 1st order models -.-.-.

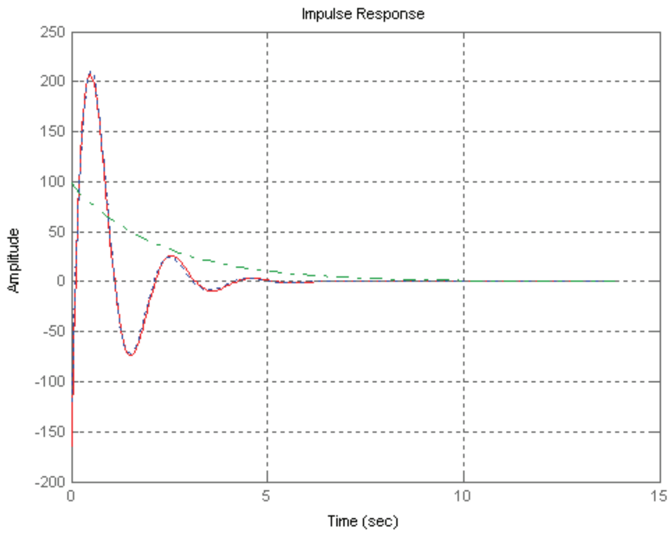

Figure 2. Impulse response of the full order model -----, reduced 2nd order ..... and 1st order models -.-.-.

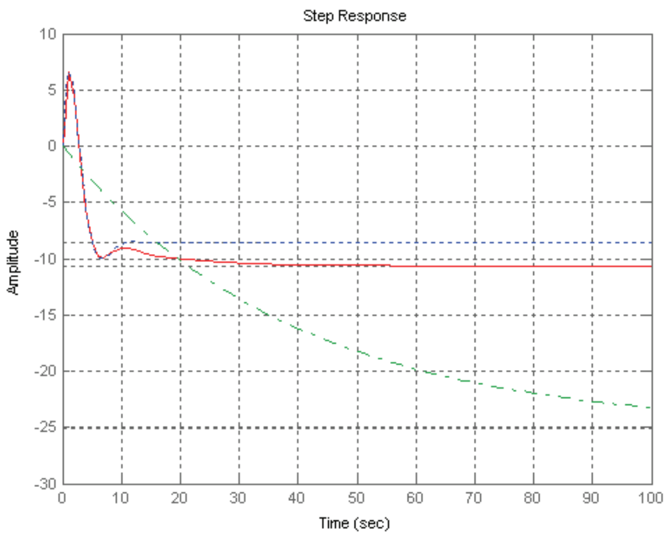

Figure 3. Step response of the full order model -----, reduced 2nd order ..... and 1st order models -.-.-.

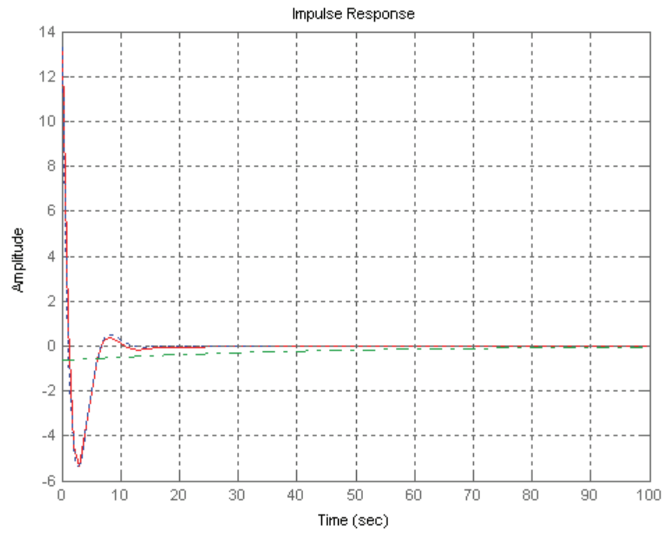

Figure 4. Impulse response of the full order model -----, reduced 2nd order ..... and 1st order models -.-.-.

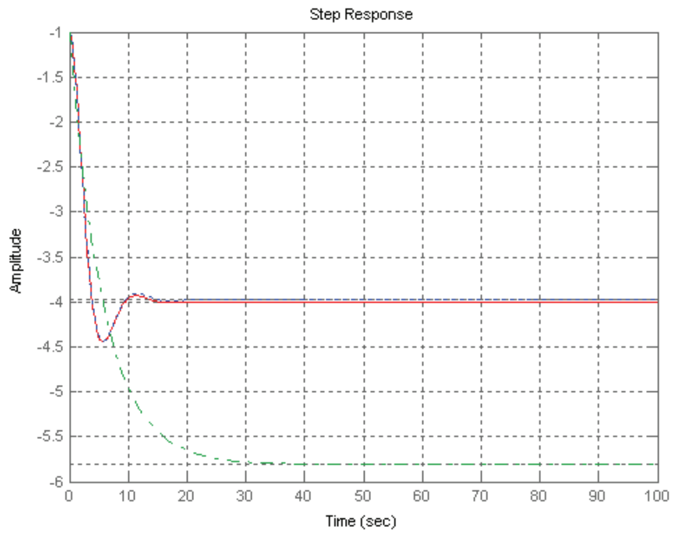

Figure 5. Step response of the full order model -----, reduced 2nd order ..... and 1st order models -.-.-.

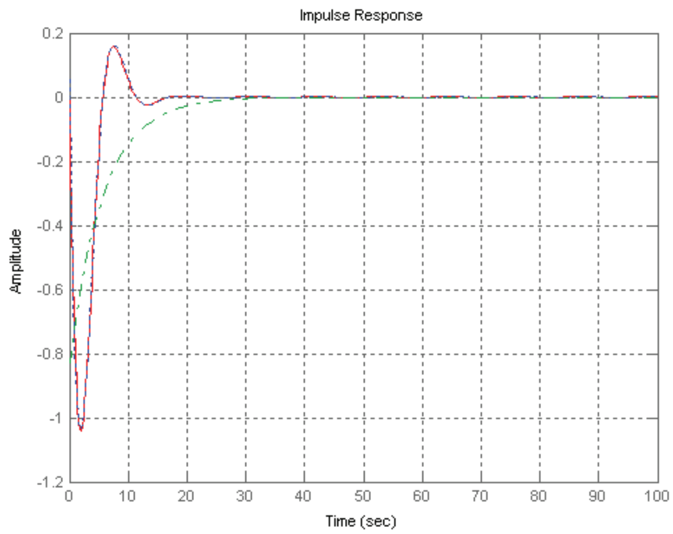

Figure 6. Impulse response of the full order model -----, reduced 2nd order ..... and 1st order models -.-.-.

description represented by a combination of differential and algebraic equations. This canonical form clearly reflects the physical substance of singular systems: the mixed dynamics and statics. The proposed direct method combines the residualization and truncation approaches for model order reduction. The residualization approach based on singular perturbation approximation is a natural approach here because the algebraic equations can be considered as constraints on the approximated fast variables. Once the fast variables are substituted into the dynamical part, the obtained regular dynamical system is approximated by using the balanced truncation approach. Different cases depending on the singularity of matrix A_{22} are considered. If matrix A_{22} is nonsingular the residualization approach does not introduce any error in approximating the system. If A_{22} is singular, the Moore – Penrose generalized inverse is computed and used in the first stage of the proposed algorithm.Finally, the case when A_{22} is the zero matrix is also considered. Different experiments are performed for all three cases showing good approximation properties of the proposed method.

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Manuscript received on 22.10.2012

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